Patterns in Patterns
Looking at the World with Math

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0 Some geometry and algebra

0.1 Triangles

0.1.1 Basics

Any triangle in a flat plane is completely determined by the lengths of its sides. Do you see why this is so? Say you want to make a triangle whose sides have lengths A, B, C. Also, assume that $A \leq B \leq C$. Start with the *longest* segment—the one whose length is C. At one of the ends you want to attach segment B and at the other segment A. Of course, one endpoint of the B segment has to be at an end of the C side, while one endpoint of the A segment is at the other end. What possible choices are there for the other endpoint of the B side? Of the A side? Make a sketch that illustrates this. Once you've selected values for A, B, and C, what are the possibilities for an (A, B, C) triangle? Does every (A, B, B) triangle have the same size and shape?

Exercises

1. Use this way of thinking about triangles to construct a:

   (a) $(2, 3, 4)$ triangle

   (b) $(2, 3, 5)$ triangle

   (c) $(1, 2, 3)$ triangle

   (d) $(\sqrt{2}, \sqrt{3}, \sqrt{5})$ triangle

2. Can you make a triangle with *any* three side-lengths you want? If not, what requirement must the lengths satisfy?

3. Is a triangle completely determined by its angles? In other words, if you know the triangle's three angles, do you know both its size and shape? Explain your answer.

4. Certainly, a triangle can't have three arbitrarily chosen angles. What rule must the angles obey? Draw a picture to help explain this?

5. What relationships are there between a triangle's sides and angles? If triangle (A, B, C) has angles a, b, c where side A is opposite angle a, etc. and $A < B < C$, what can you say about the angles?

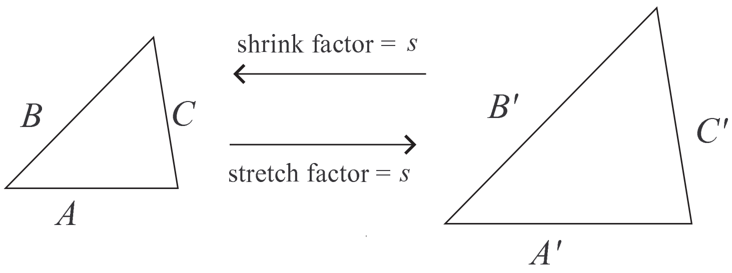
6. An "isosceles" triangle has two equal sides. What can you say about the angles opposite the two equal sides?

0.1.2 Similar triangles

Two triangles are similar if they have the *same shape*. Geometrically, this means that they have the same angles. If you have two similar triangles, each side of one is "paired up" with a side of the other. What determines how they pair-up?

Two similar triangles are just "scaled up" or "scaled down" versions of each other. Let's look at what this means. Consider two similar triangles (A, B, C) and (A', B', C') where side A corresponds to A', B corresponds to B', and C corresponds to C'. There's a stretching (or shrinking) of (A, B, C) that makes it the same size as (A', B', C'). In terms of algebra, there's a *stretch-factor* (or *shrink-factor*) s such that

$$s\,A = A' \qquad s\,B = B' \qquad s\,C = C'.$$



One of the great things about similar triangles (and similar objects, in general) is that the *ratios of corresponding sides are equal.*

Exercises

1. Express the ratios-of-corresponding-sides-are-equal property *symbolically* for the (A, B, C) and (A', B', C') triangles. Why are the three ratios equal?

2. If triangle $(2, 3, 4)$ is similar to $(6, 9, x)$, what is x?

3. If triangle $(2, 5, 6)$ is similar to $(6, x, y)$, what are x and y?

4. Are the following pairs of triangles similar? If they are, what's the scale factor? If they're not, how can you tell?

   (a) $(2, 5, 6)$ and $(3, 7, 8)$
   (b) $(2, 6, 7)$ and $(3, 9, 10)$
   (c) $(2, 6, 8)$ and $(3, 9, 12)$
   (d) $(\sqrt{3}, \sqrt{5}, \sqrt{7})$ and $(\sqrt{6}, \sqrt{10}, \sqrt{14})$.

0.1.3 Right triangles

A "right triangle" has a right angle for one of its angles. Can a triangle have two right angles? One of the most special of all mathematical relationships is expressed by the **Pythagorean Theorem**. This says that for *any* right triangle (A, B, C) whose hypotenuse—the triangle's longest side—is C,

$$A^2 + B^2 = C^2.$$

This works the other way as well: if a triangle (A, B, C) satisfies

$$A^2 + B^2 = C^2,$$

(A, B, C) is a right triangle whose hypotenuse is C.

Exercises

1. Sketch an "arbitrary" right triangle (A, B, C) and from the vertex at the right angle draw a segment to the hypotenuse that is also perpendicular to it. This is called the "altitude" or "height" of the triangle. Call it H and label it in your picture. It creates two triangles *within* the original one. What can you say about these two triangles? Pay attention to the angles.

2. Use the properties of similar triangles to derive the Pythagorean equation. You'll need to introduce a new label for a side of one of the "inside" triangles. Call it D. Now write down all of the equal ratios that you see and try to eliminate everything but A, B, C.

3. What's x, if

   (a) $(3, 4, x)$ is a right triangle?

   (b) $(5, 12, x)$ is a right triangle?

   (c) $(\sqrt{3}, 2, x)$ is a right triangle?

4. Are the following right triangles?

   (a) $(5, 7, 10)$

   (b) $(\sqrt{5}, \sqrt{7}, \sqrt{12})$

   (c) $(\sqrt{7} - \sqrt{2}, \sqrt{7} + \sqrt{2}, 3\sqrt{2})$

5. How can you use the Pythagorean theorem to determine whether or not a triangle has an angle that's bigger than $90°$.

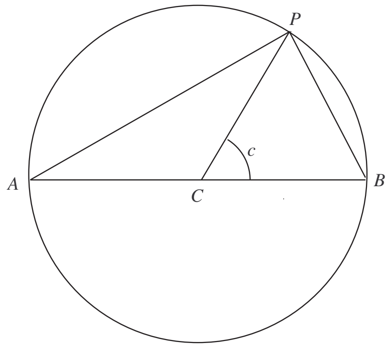
6. Is the center of the pitcher's mound in the *center* of the infield "diamond." (It's actually a square)? Note: 1st base is 90 feet from 2nd base and home-plate, 2nd base is 90 feet from 1st and 3rd bases, and 3rd base is 90 feet from 2nd base and home-plate. The center of the pitcher's mound is 60 feet 6 inches from home-plate on the line from home-plate to second base.

7. **Triangle in a circle** Take a circle with radius one and select a diameter that hits the circle in points A and B. Now, pick *any* point P on the circle other than A or B and connect A and B to P.

   (a) Just by looking, guess what the angle at P is.

   (b) Now let's show that this has to be a right angle. This must have something to do with the circle. So, let's use the fact that A, B, and P are on the circle. Call the center of the circle C. What can you say about the sides and angles of the triangles CAP and CBP?

   (c) Call one of the angles at C by the name c. Express all the other angles in the two triangles in terms of c. Finally, derive that the angle at P is $90°$.



0.2 π

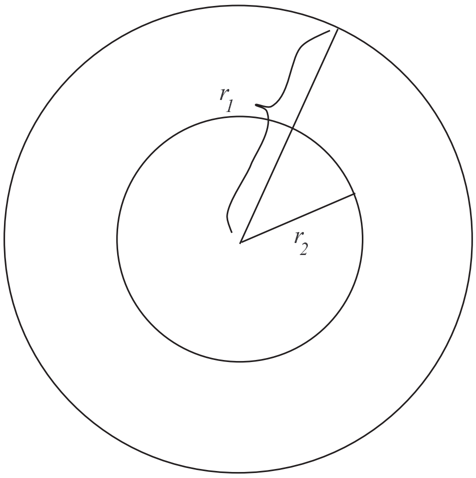
Select a point C in the plane. The set of points that are a fixed distance r away from C is called *the circle of radius* r *and center* C. Suppose you have two circles of radii r_1 and r_2 where $r_1 > r_2$ and circumferences C_1 and C_2. Stretching the smaller circle so that it's the same size as the larger *means* that there's a stretch factor s with

$$r_1 = s\,r_2 \qquad \text{and} \qquad C_1 = s\,C_2.$$

This implies that

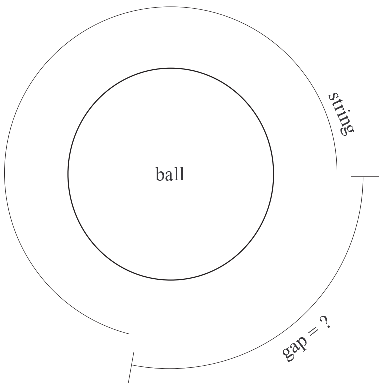
$$\frac{C_1}{r_1} = s = \frac{C_2}{r_2}.$$

Now, this tells us that this ratio does not depend on the size of the circle. We can thereby *define* the ratio of *any* circle's circumference to its diameter as a certain number. We call this number π.



Exercises

1. Suppose that you wrap a string around a basketball and then lift the string one foot directly above the ball's surface. This opens up a gap. How much string do you need to fill in this gap? What happens if you do the same thing on the surface of the earth? How much string does it take to fill the gap?



0.3 Measuring angles—degrees and radians

The *number* π is the *ratio* of a circle's circumference to its diameter. The amazing thing is that this ratio does not depend on the *size* of the circle. Actually, we can think of this in terms of similar shapes. Given two circles whose diameters are 2 and 3, we stretch the smaller circle by a factor of 3/2 so that it's the same size as the bigger one. The circumference also stretches by 3/2 so that

$$\frac{3}{2}\,(\text{circumference of smaller circle}) = \text{circumference of smaller circle}$$

and

$$\frac{\text{circumference of smaller circle}}{\text{diameter of smaller circle}} = \frac{\text{circumference of larger circle}}{\text{diameter of larger circle}}\,.$$

A circle of radius r has circumference $2\pi r$ and when $r = 1$, the circumference is 2π. This allows us to use the distance *along* an arc of the circle to measure an angle made by two rays at the circle's center. Now, take a bunch of pieces of string each of whose length is that of the radius. If you laid the pieces of string *along* the circle, you would need $2\pi \approx 6.25$ of them to go all the way around. Each piece "covers" an angle known as a *radian.* So, there are 2π radians around a circle. Since there are also $360°$ around a circle,

$$2\pi \text{ radians} = 360°.$$

One more thing: let's agree to start measuring angles at the *horizontal* ray that goes to the right and go around the circle *counterclockwise* to measure a *positive* angle and *clockwise* to measure a *negative* angle. Using radians instead of degrees to measure angles is like using meters rather than feet to measure distance.

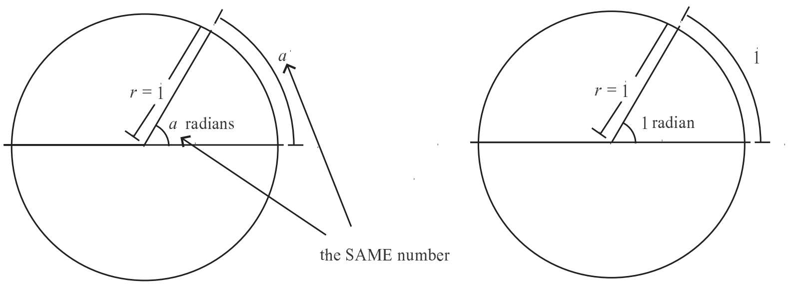


Figure 1: The great thing about radians.

Exercises

1. How many degrees are there in:
   (a) *one* radian?
   (b) $\frac{\pi}{4}$ radians?
   (c) π radians?

2. How many radians are there in:
   (a) *one* degree?
   (b) 90 degrees?
   (c) 135 degrees?

0.4 Quadratic equations

A *quadratic equation* has the form

$$a\,x^2 + b\,x + c = 0.$$

The expression on the left is called a *polynomial of degree* 2. Some examples are

$$x^2 - 1 = 0$$
$$x^2 + 1 = 0$$
$$2\,x^2 - 3\,x + 1 = 0$$
$$\sqrt{2}\,x^2 - 6\,x - \pi = 0.$$

0.4.1 Solving an equation

A *solution* to an equation $a\,x^2 + b\,x + c = 0$ is a *number* s that *satisfies* the equation. This means that when x is replaced by s the equation is true (note that an equation is a *statement*):

$$a\,s^2 + b\,s + c = 0.$$

Exercises

Why or why not?

1. Is 3 a solution to $x^2 - 3 = 0$?
2. Is $3\sqrt{3}$ a solution to $x^2 - 18 = 0$?
3. Is 1 a solution to $3\,x^2 - 2\,x - 1 = 0$?
4. Is -1 a solution to $3\,x^2 - 2\,x - 1 = 0$?

0.4.2 Factoring

Equations like this come up quite often when you want to describe things like triangles and spirals. In fact, you've already seen them in the section on right triangles.

What we'd like to to do is *solve* such an equation. This means finding numbers that make the equation true when they're substituted in for x. The simplest approach is by means of factoring. If you could express the equation in terms of a product of two simpler things, you could just "read off" the solution.

For instance, since you can "factor" $x^2 - 1$ into the product

$$(x-1)(x+1),$$

the solutions to

$$x^2 - 1 = 0$$

are

$$x = 1 \qquad \text{and} \qquad x = -1.$$

Make sure that you understand why this is.

Exercises

Try to factor the following equations. If you can, what are the solutions?

1. $x^2 - 4 = 0$
2. $x^2 + 2\,x + 1 = 0$
3. $2\,x^2 - 5\,x + 2 = 0$
4. $2\,x^2 - 4\,x + 2 = 0$
5. $x^2 + x + 1 = 0$

It may happen that you don't see an easy way of factoring the polynomial. We need a more robust method.

0.4.3 The quadratic formula

If we don't see how to factor, let's try to put a quadratic equation into a form that will allow us to see what the solutions are. We could do this if we had a way of getting rid of the $b\,x$ part of the equation. This would give us an equation *like*

$$a\,x^2 + c = 0$$

which is easily solved.

A specific equation

1. What are the solutions of

$$3\,x^2 - 2 = 0?$$

2. Express *in words*: a is the square-root of b means ....

3. Substitute $x = y + d$ into the equation

$$2\,x^2 - x - 2 = 0.$$

(Think of d as a number whose value you don't yet know.)

4. Manipulate the result until you get an equation that "looks like" (has the *same form* as)

$$a'\,y^2 + b'\,y + c' = 0.$$

In terms of d, what are a', b' and c'.

5. What value must d have in order for b' to be 0? Let d have this value. What does the equation become? Is it easy to solve?

6. Solve this equation for y and then get solutions to the original equation by using

$$x = y + d.$$

A general equation

1. Substitute $x = y + d$ into the equation

$$a\,x^2 + b\,x + c = 0.$$

(Think of d as a number whose value you don't yet know.)

2. Manipulate the result until you get an equation that "looks like" (has the *same form* as)

$$a'\,y^2 + b'\,y + c' = 0.$$

In terms of a, b, c and d, what are a', b' and c'.

3. What value in terms of a, b, c must d have in order for b' to be 0? Let d have this value. What does the equation become? Is it easy to solve?

4. Solve this equation for y and then get solutions to the original equation by using

$$x = y + d.$$

Exercises

Use the *quadratic formula* that you found above to solve the following equations.

1. $x^2 + 3\,x + 1 = 0$

2. $2\,x^2 - 5\,x + 1 = 0$

3. $x^2 - x - 1 = 0$

4. $x^2 + x + 1 = 0$.

1 Symmetry

Most of us have some idea or sense of what symmetry is. But, have you ever tried to describe it carefully and precisely? What does it mean for a painting, sculpture, building, shape, . . . to be *symmetric*? Is the human body symmetric? *Exactly* symmetric? What do you notice when you see *that* something is symmetric? Let's begin with simple cases.

1.1 Polygons: symmetric shapes in two dimensions

Take a *regular* (equilateral) triangle. If you don't watch, can I somehow move the triangle so that, when you look at it again, you can't tell that I moved it? Moving the triangle this way is called a *symmetry* of the triangle.

The center of the triangle is special. If I apply a symmetry, the center point can't move. Otherwise, you could tell that I've moved the triangle. We'll call a point that doesn't move when a symmetry is applied a *fixed point*. What kind of motion can I apply to the triangle that *doesn't move* its center? Rotations come to mind—spinning an object about some point doesn't move that point. So, maybe we can rotate the triangle about its center. By how much (what angle) do I turn in order for the triangle to *look* the same after I've turned it? The corners are also special; a symmetry must move one corner to one of the others—maybe to the *same* corner. Notice that the corners are *not* fixed points of the rotation. How many of these turns can I make before I'm back where I started? We'll count doing *nothing* to the triangle as a symmetry. In this sense, everything has some symmetry.

Can you think of any symmetries of the triangle that aren't rotations? (Hint: What do mirrors do to an object? Can you divide the triangle into two pieces that are mirror images of each other? Draw pictures.)

To describe a rotational symmetry you must specify a *center of rotation* and an *angle of rotation* (the amount by which you rotate). To describe a reflective symmetry, you must specify a *line of reflection*—a mirror, in effect.

Opportunities

1. State a general definition of symmetry. ("An object is *symmetric* when ....) Like the U.S. Constitution, your definition should apply to a wide variety of cases. To do so, you might want to make it somewhat vague.

2. Describe and count the symmetries of a square. First restrict yourself to rotations and then consider reflections (mirror image symmetry). How many symmetries are there? Describe the fixed points for each one.

3. Describe and count the symmetries of a regular pentagon. First restrict yourself to rotations and then consider reflections. How many symmetries are there? Describe the fixed points for each one.

4. Describe and count the symmetries of a regular polygon with 17 sides.

5. Describe the rotational symmetries of the upper-case English letters. (Use the font of Figure 2.) Put letters that have the *same symmetries* into groups. For example, H and X have the same rotational symmetries—what's the center and by how much do you turn? (There might be others of the *same type*.)

6. Describe the reflective symmetries of the upper-case English letters. Put letters that have the *same symmetries* into groups. For example, A and T have the same reflective symmetries—where do you divide the letters to make two mirror images.

A B C D E F G H I J K L M
N O P Q R S T U V W X Y Z

Figure 2: What are the symmetries?

7. a) If you pay attention to the shading (that is, distinguish the white parts from the black parts), what are the symmetries in Figure 3?

   b) If you *ignore* the shading (that is, **don't** distinguish the white parts from the black parts), what are the symmetries in Figure 3?

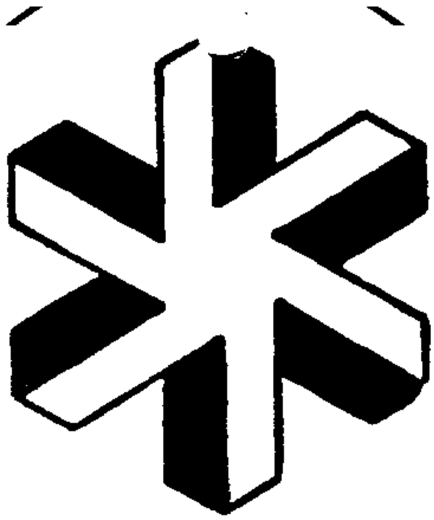


Figure 3: Pinwheel

1.2 Tiling

One common occurrence of symmetry is in floor tiles. Use the snap together "polydrons" to determine if you can cover a floor with equally-sized regular triangles so that *no gaps* appear and the pattern extends *indefinitely* in all directions. For obvious reasons, this is called a *regular tiling* or *tessellation* of a plane. In general, a *plane-tiling* is a gapless pattern of flat shapes—not necessarily polygons.

Opportunities

1. Now, try to decide whether you can make a tiling of a plane with squares and rectangles in general.

2. What about a regular tiling with pentagons? Hexagons (six-sided)? Heptagons (seven-sided)? If you can't do it, try to describe what the problem is.

3. Create a tiling of a plane where each tile is a triangle that's **not equilateral**. What things must you take into consideration in designing your tiles? How can you make sure that they fit together with no gaps and can extend indefinitely in all directions? What symmetries do your *individual* tiles have? What about the symmetry of the tiling *as a whole*? (Remember what we mean by 'symmetry.')

4. Try to tile the plane using *two different* polygons. Examine your tiling for *overall* symmetry.

1.3 The essence of a tiling

When looking for the symmetry of a tiling—not just the individual tiles—we look for things that we can do to the tiling that does not change its appearance. Look at the square tiling in Figure 4. How can we "move" the tiling so that it looks the same? There are three *different* reflective symmetries. Notice that the ones chosen in the figure enclose a right triangle (shaded). The vertices of this triangle are at three *different* types of point on the square tile: a corner, a mid-point of an edge, the center of the square.

Now, let's use the symmetries of the tiling to move the shaded triangle—call it F_0. When F_0 is reflected over mirror 3 a new triangle just like F_0 appears. If we put F_0 and its mirror image together, we get yet another triangle (Figure 5). Call it F_1. Now, let's reflect F_1 across mirror 1. What happens? Another triangle—call it F_2 (Figure 6). Finally, when F_2 is reflected across mirror 4, we get the whole square (Figure 7). What now? Reflection of the square across mirror 2 produces a rectangle (make the corresponding sketch). You should convince yourself that if we continue to apply symmetries of the tiling, we can make the shaded part grow to fill up as much of the tiling as we want. Can you find some non-reflection symmetries that help you see this?

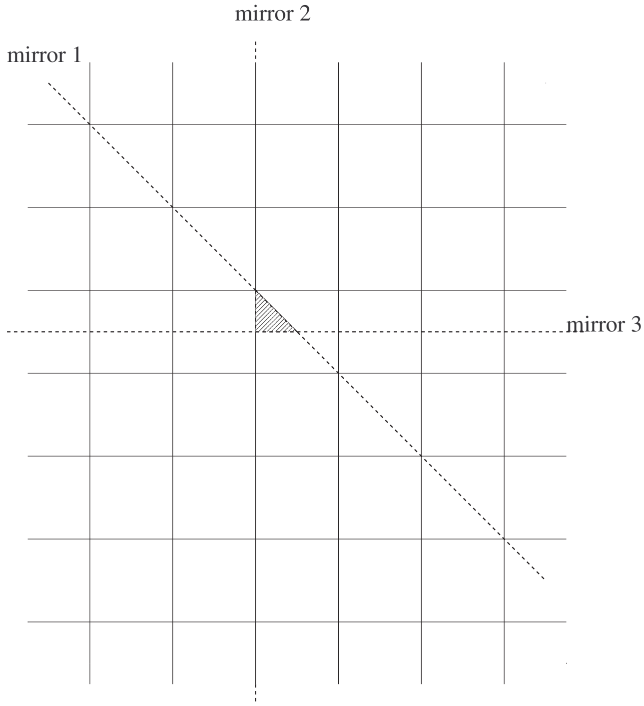


Figure 4: Reflective symmetries of the square tiling and the part F_0

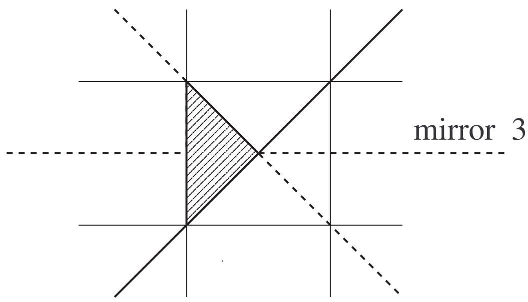


Figure 5: Getting F_1 from F_0: reflect across mirror 3

Remember that this started from the "small" triangular piece F_0 that is one-eighth of a square. Could we have begun with a smaller piece? There are three things to notice here:

1. Applying the tiling's symmetries just to F_0 generates the whole tiling.

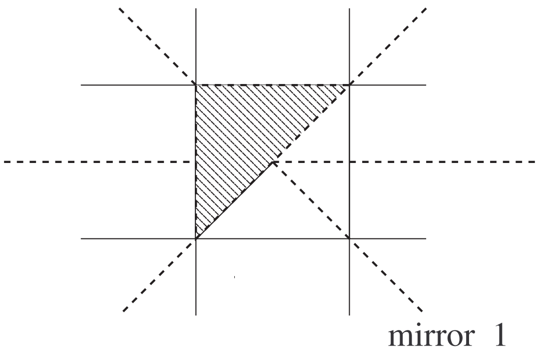


Figure 6: Getting F_2 from F_1: reflect across mirror 1

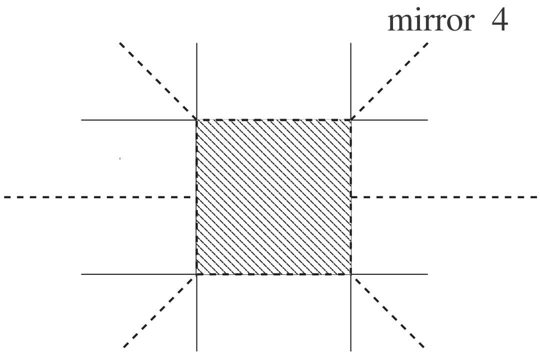


Figure 7: Getting the square tile from F_2: reflect across mirror 4

2. The only symmetries that moves F_0 *onto itself* is the trivial (do-nothing) symmetry.

3. Any *non-trivial* symmetry moves F_0 *entirely* away from where it was—there's no overlap between F_0 and the region where F_0 moved to.

Thus, *no smaller piece* of F_0 can be used to create the *entire* tiling. We'll call a part of a tiling with this property a *generating part.*

Opportunities

1. We know that a generating part can have a mirror on its outside border. Can a generating part have a mirror passing *through* it (that is, on the

inside)? If not, why not? If so, what's an example?

2. We know that a generating part can have a center of rotation on its outside border. Can a generating part contain a center of rotation on its inside? If not, why not? If so, what's an example?

3. Find a generating part of the following tilings. In each case show how to genreate the whole tiling by moving the generating part according to the *tiling's* symmetries. Also, why do you think that no smaller part will work?

   (a) the regular triangular tiling

   (b) your tilings from Section 1.2.

   (c) the tilings in Figure 8.

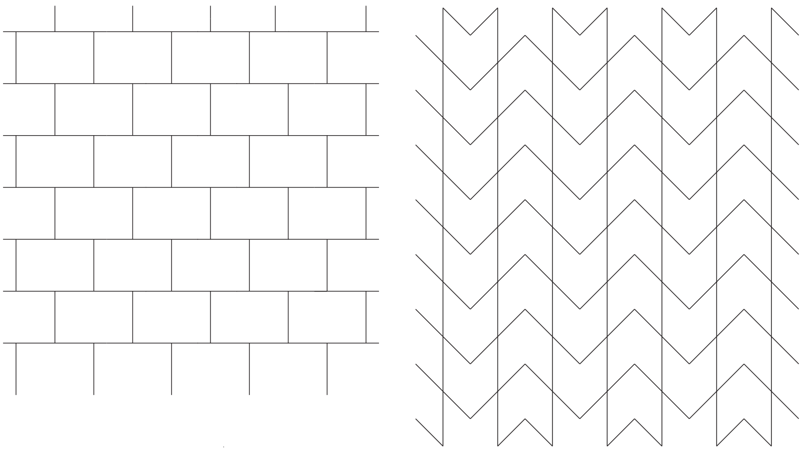


Figure 8: Find a generating part

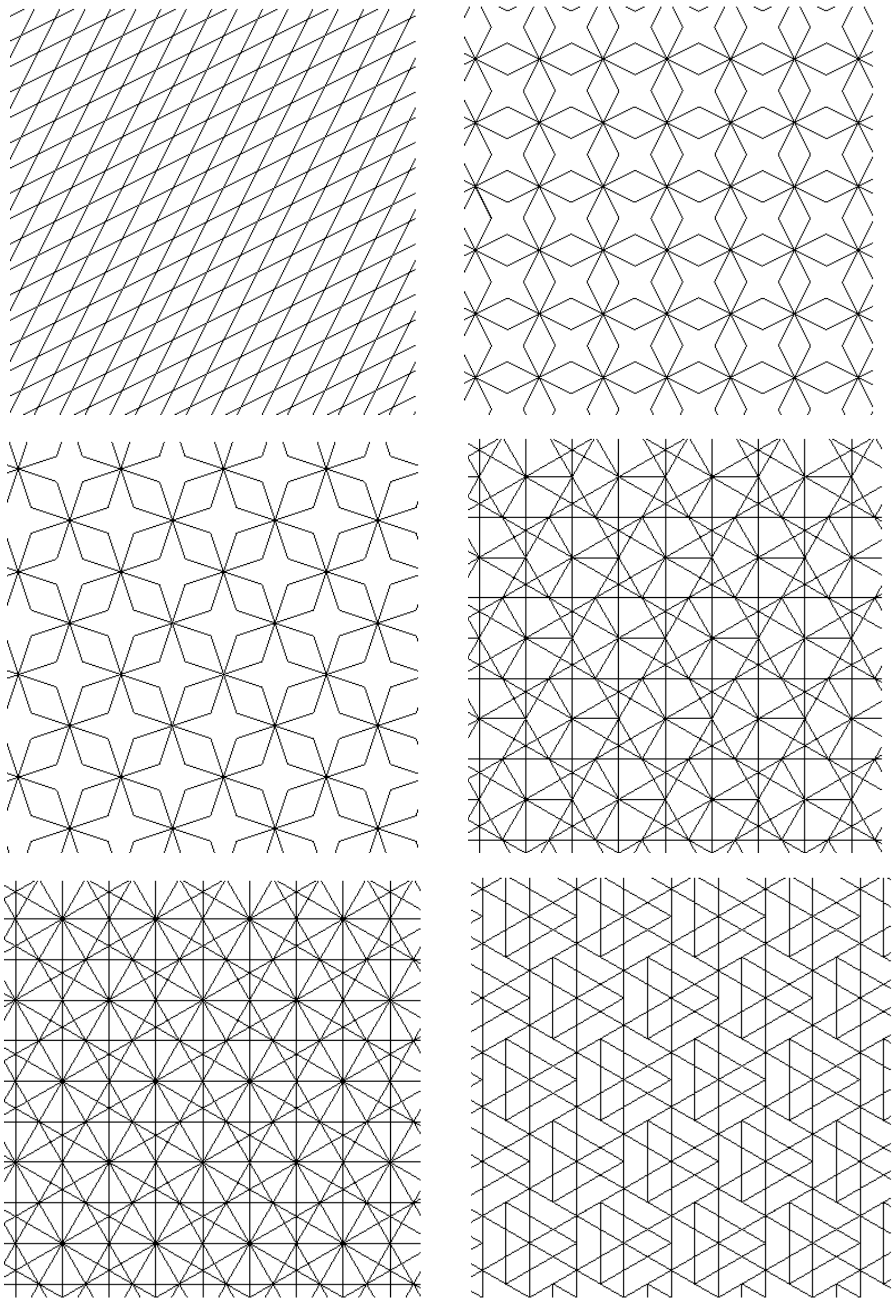


Figure 9: Find a generating part

1.4 Polyhedra: symmetric shapes in three dimensions

In a two dimensional world houses might be built in the shape of polygons. To completely close up, such a house would have at least three walls. Would it have a floor and ceiling?

There's more going on in three dimensions. What's the smallest number of enclosing surfaces—walls, ceiling, floor—that a house can have here? Using only regular triangles, make an object that closes up and has no gaps. What's the *smallest* number of triangles that you can do this with? What if you use squares?

A shape of this kind is called a *polyhedron* meaning "many faces." The polygons that make up the polyhedron are called *faces*. The other distinguished parts are the *edges* and *vertices*. For the two polyhedra that you made, count the number of vertices (V), edges (E), and faces (F). We're looking for patterns in these numbers. Do you see any similarities or connections between the two polyhedra or their respective sets of VEF numbers?

1.4.1 Plato's perfect polyhedra

Consider the two polyhedra that you've made. Of course, there are always two faces at an edge. How many faces surround a vertex? Does the number of faces at a vertex change depending upon which vertex you consider? Polyhedra like these two—each face is a regular polygon and each vertex has the *same* number of faces around it—are called *regular*. Plato ascribed cosmological significance to such shapes. Just how many of these perfect objects do you suspect there are?

Since a regular polyhedron has the same number of faces at each vertex, we could start joining a certain number of one type of regular polygon to form cone-like structures. Then we could try to connect these structures to get a full-blown polyhedron.

Opportunities

1. The simplest polygon we could use is a regular triangle. You've already made one polyhedron out of triangles. How many triangles meet at a vertex? Is this the smallest number of faces that can surround a vertex? How about putting in one more triangle at a vertex? Try it. Can you see how to use the resulting cone-structure to make a *regular* polyhedron? Count its V, E, and F.

2. How about putting in another triangle at *each* vertex? Use cone-structures with four triangles to build a regular polyhedron. This will take some playing around. Again, what's V, E, F for this object?

3. Do you think that we can just keep going putting in triangles at a vertex? Try putting in another one. What happens? Can you use this to construct a polyhedron? What if you put in yet another triangle? Make a conjecture about building regular polyhedra with triangles.

4. Take a look at the polyhedron that you made with squares. How many faces at a vertex? Can you put in more squares at a vertex? What goes wrong? Make a conjecture about building regular polyhedra with squares.

5. The next regular polygon to consider is the pentagon. Use the same approach as above to decide whether there are any regular polyhedra made of pentagons. Make a conjecture about building regular polyhedra with pentagons.

6. Now consider using regular hexagons. Heptagons? What's happening here? You can think of a regular polyhedron as a *tiling* of a sphere—imagine that the faces are balloon material and blow air into it. How many regular tilings of a sphere do there seem to be? Compare this to what we discovered about regular tilings of a plane.

1.4.2 V, E, and F

You should have built models of five regular polyhedra and conjectured that there are no others. You might want to make paper models of these. If so, try laying out the whole thing on a flat sheet of paper that you will cut, bend, and glue.

- Now, make a table for these showing the number of vertices, faces, and edges. Look for similarities between them. Experiment with the numbers in order to detect patterns.

- Build a non-regular polyhedron (any way you can—use different types of polygons for faces, if you like) and record its VEF. Why does you polyhedron *fail* to be regular? Compare its VEF to the numbers that you already have. Anything?

1.4.3 Polyhedral symmetry

For the regular tetrahedron and the cube, count and describe the rotational symmetries that it has. (Bear in mind what you must specify when describing a rotation.) A good idea here would be to begin with the simplest one and try to develop a way of counting that makes use of the object's symmetry. Test it out on the next simplest shape. Are there patterns among the symmetries?

Opportunities

1. Try to construct a polyhedron that has a hole like a donut. Count VEF and compare to the previous results.

2. Find a fundamental part for each regular polyhedron using *just rotational* symmetries. (Paper models might be useful here.) What happens if you allow reflections as well?

3. Describe the symmetries of a baseball. (Ignore the stitching and think of the seam as a curve.) Build a polyhedron that has *precisely* the same symmetries as a baseball.

   Suggestion: Look for a simple shape. Think of edges as corresponding to the seam.

4. Describe the symmetries of the regular octahedron, dodecahedron, and icosahedron.

2 Spirals

2.1 Spinning a web

Opportunities

1. On a sheet of "octopus" paper, mark a point (about one inch from the center) on one of the rays emanating from the center. Label the point P_0.

2. *Carefully* place an edge of an index card (or something comparable) *along* the ray containing P_0 so that a corner of the card is at P_0. Notice that there are *two ways* of doing this. Pick the one for which the card lies on the "counter-clockwise side" of the ray. Now along the other edge that hits P_0 draw the segment from P_0 to the *next* ray. Call the point that you reach on the next ray P_1.

3. Repeat the process at P_1 to obtain a point P_2 on a third ray (not the one that contains P_0). Repeat at P_2 to get P_3. Continue this "iterative" or "recursive" procedure until the rays aren't long enough to get a new point.

4. In general terms describe the "curve" that you've produced? What *causes* this to happen? What would happen to the picture if there were *more* equally-spaced rays around the center?

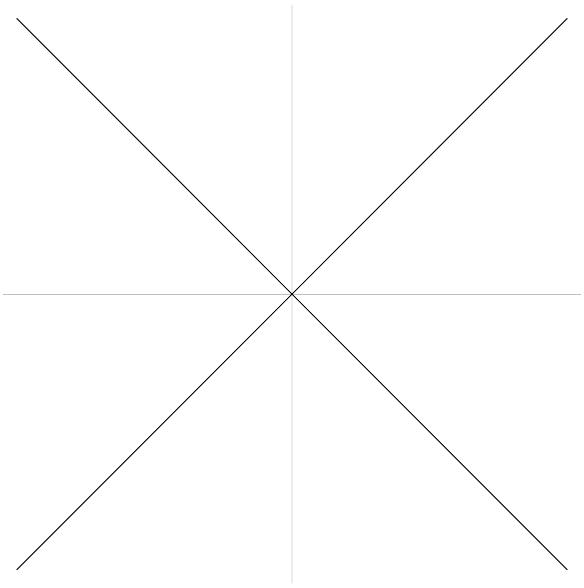
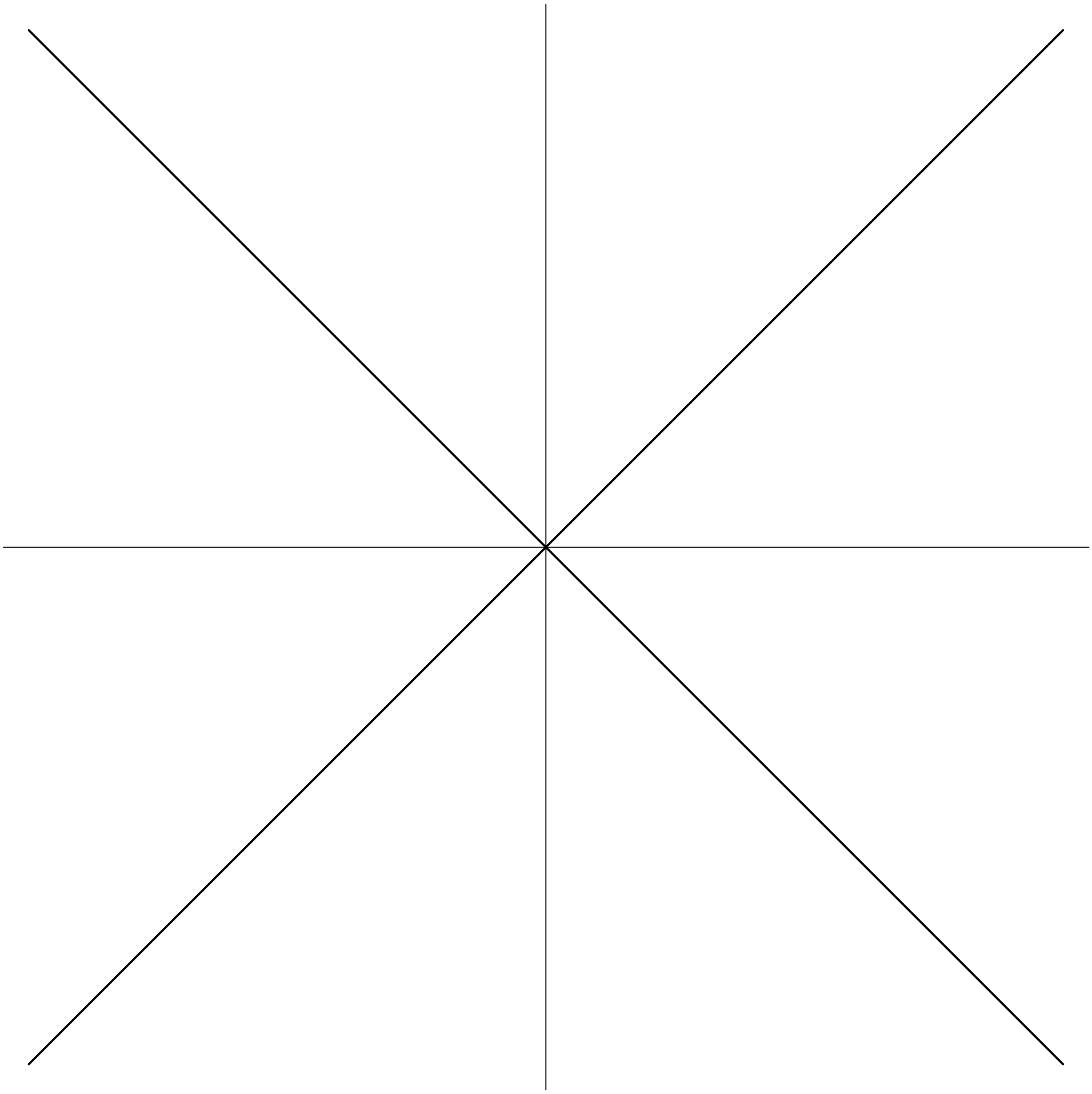
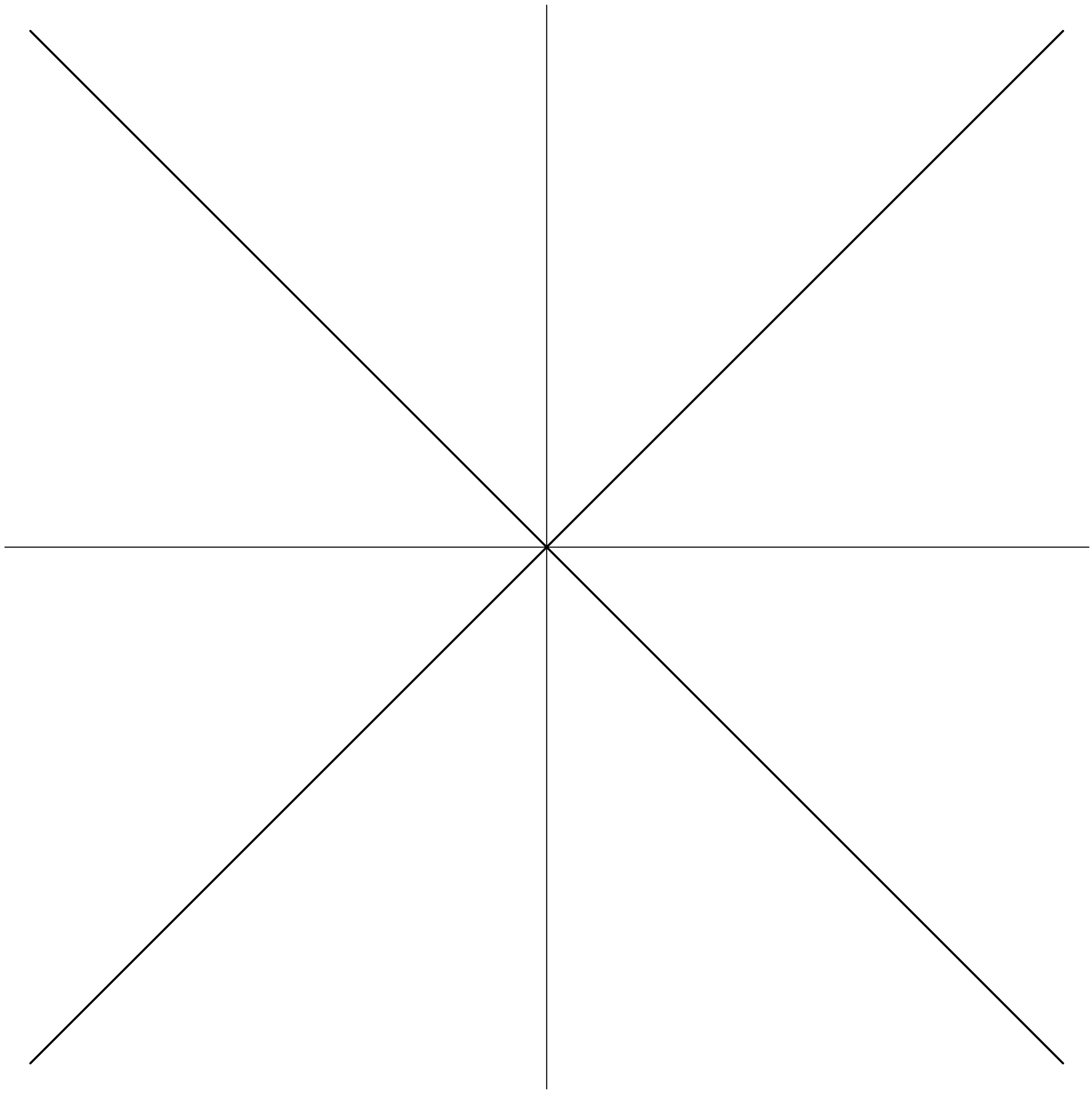


Figure 10: Octopus paper





2.2 Continuing the spiral

Opportunities

1. Let's extend the spiral in the "other" direction. Place one edge of the index card at P_0 and the other edge *along the first* ray in the direction *opposite* to your spiral (away from P_1). Mark the segment from P_0 to the place where the corner meets the ray—call that point P_{-1}. As before use d_{-1} to specify the distance from C to P_{-1}.

2. Now, apply this procedure to the point P_{-1} to obtain a point P_{-2}. Repeat this at P_{-2} to get P_{-3}, at P_{-3} to get P_{-4}, . . . , until you can't continue.

3. What's the effect of this? If you continued the process indefinitely or infinitely, what would happen? If you were to look at the part of the spiral near C with a microscope, what would you see? What happens as you strengthen the microscope's magnification? Does this shed any light on why a snail or nautilus makes this kind of spiral?

2.3 The spiral formula

Opportunities

1. In the same way that we found a relationship between d_1 and d_2, d_3, . . . , express d_{-1} in terms of d_0. How about d_{-2}, d_{-3}, . . . ? What's the pattern?

2. Call the ray that contains P_0 the $0°$ (radian) ray. That is, measure angles *starting* at the P_0 ray. What angle do two consecutive rays form?

3. Make a table that shows the angle of a ray and the distance from C to the spiral point *on that ray*. Do this for five rays. Do you see a relationship between the angle and the distance? Call the angle t and the distance to the spiral point r. Express r as a function of the angle t.

4. If you didn't know that $d_0 = 1$, how would you express the distance to the spiral points? How would things change in this case? Try using a specific value, say $d_0 = 3$.

5. This formula allows you to "fill in" a smooth spiral between rays. According to this formula, how far away from C is the point on the smooth spiral that corresponds to an angle of $30°$ $(\pi/6)$? $-60°$ $(-\pi/3)$?

Illustrate these points on the octopus paper. In a color different from your "segmented" spiral carefully draw in a smooth spiral that passes through all of the spiral points. Compare the angles made where the rays cross the smooth spiral.[1] Without computing a specific numerical value, what can you say about these angles? Use the idea of similarity to justify your answer.

2.4 Continuing the spiral

Opportunities

1. Let's extend the spiral in the "other" direction. Place one edge of the index card at P_0 and the other edge *along the first* ray in the direction *opposite* to your spiral (away from P_1). Mark the segment from P_0 to the place where the corner meets the ray—call that point P_{-1}. As before use d_{-1} to specify the distance from C to P_{-1}.

2. Now, apply this procedure to the point P_{-1} to obtain a point P_{-2}. Repeat this at P_{-2} to get P_{-3}, at P_{-3} to get P_{-4}, ..., until you can't continue.

3. What's the effect of this? If you continued the process indefinitely or infinitely, what would happen? If you were to look at the part of the spiral near C with a microscope, what would you see? What happens as you strengthen the microscope's magnification? Does this shed any light on why a snail or nautilus makes this kind of spiral?

2.5 The spiral formula

Opportunities

1. In the same way that we found a relationship between d_1 and d_2, d_3, ..., express d_{-1} in terms of d_1. How about d_{-2}, d_{-3}, ...? What's the pattern?

2. Call the ray that contains P_0 the $0°$ (radian) ray. That is, measure angles *starting* at the P_0 ray. What angle do two consecutive rays form?

3. Make a table that shows the angle of a ray and the distance from C to the spiral point *on that ray*. Do this for five rays. Do you see a relationship between the angle and the distance? Call the angle t and

[1] The spiral is curved. You have to decide what it means for *curves* to intersect at an angle.

the distance to the spiral point r. Express r as a function of the angle t.

4. If you didn't know that $d_0 = 1$, how would you express the distance to the spiral points? How would things change in this case? Try using a specific value, say $d_0 = 3$.

5. This formula allows you to "fill in" a smooth spiral between rays. According to this formula, how far away from C is the point on the smooth spiral that corresponds to an angle of $30°$ $(\pi/6)$? $-60°$ $(-\pi/3)$? Illustrate these points on the octopus paper. In a color different from your "segmented" spiral carefully draw in a smooth spiral that passes through all of the spiral points. Compare the angles made where the rays cross the smooth spiral.[2] Without computing a specific numerical value, what can you say about these angles? Use the idea of similarity to justify your answer.

6. By how much can you stretch or shrink the spiral so that it appears to be unchanged—lands on itself? Give the specific scale-factor.

2.6 The coil

Imagine a tightly-wound rope that's lying flat on a floor—it forms a spiral.

1. Make a sketch of this spiral. As you follow it inward, does the spiral *stop* or can you just keep going forever as in the case of the snail-spiral?

2. Suppose the rope is one inch thick (in diameter). Each time you wrap the rope around one complete turn, you add an inch to the spiral (measured along a ray from the spiral's center). Describe as we did before the spiral made by the inner edge of the rope: express the distance r from the center in terms of (as a function of) the turning angle t. Imagine that you're at the center of the spiral and that your dog is on a retractable leash. Assume that the dog's distance from you is 0 at the start. If you stay put and the dog walks *outward* along the coil spiral while the leash remains tight, how far away is the dog when you've made one complete turn (you've spun around once and are back in your original position)? One-half of a complete turn? One-fourth of a complete turn?

3. What happens to the angle formed by the spiral and the leash? You might think about the path the dog follows when it's very far away.

[2]The spiral is curved. You have to decide what it means for *curves* to intersect at an angle.

Now, what happens to this angle if you take your dog for a walk on the snail spiral?

2.7 Another kind of coil

Imagine that you have a rope that's in the shape of a cone. When you wind it tightly, you get a snail-like spiral. Let's see why.

1. If you have a cone that goes on forever, what happens to it if you stretch or shrink it (move *all* the points away from or toward the vertex)? By looking at the cone *as a whole*, can you tell that something has happened to it? Compare this to stretching a cylinder.

2. Apply this idea to the coiled cone-rope. If you stretch or shrink the coil by the same amount in all directions, what happens to the spiral? What happens to the snail-spiral when you stretch or shrink it in this way? Does this help you understand why a snail or nautilus makes the kind of spiral that it does?

2.8 Folding a spiral

1. Take a sheet of 8.5×11 paper and fold its *length* in half. is the resulting rectangle *similar* to the original rectangle? (Just as in the case of triangles, two rectangles are similar if ratios of corresponding sides are equal.) If it is, what's the stretch factor?

2. Now, take a roll of paper that's one unit of length wide. How long must a piece of this paper be in order for it to satisfy the above property? That is, how long of a piece will make a rectangle similar to the original when folded in half?

   Think of the length as an unknown quantity x while the width is one unit. When the length is folded in half, what are the dimensions of the rectangle that's formed? What value must x have in order for the original rectangle to be similar to the new one?

3. Use the Pythagorean theorem to figure out a way of folding the paper of width one so that the crease has the length x that's required for obtaining a rectangle that's similar to half of itself. Using such a carefully-made crease as a *ruler*, carefully cut a rectangular piece of the appropriate length. *Note: if the paper's dimensions aren't quite close to the required lengths, the following procedure won't work very well.*

4. Call the unfolded rectangle R_0. Now, mark the crease that you get when you fold the length in half. Holding the rectangle with the length on the horizontal, pick the *right* half and call it R_1. Divide R_1 in half and call the *top* rectangle R_2. How does R_2 compare to R_1? Continue this process:

   - divide R_2 in half; take the *left* rectangle to be R_3
   - divide R_3 in half; take the *bottom* rectangle to be R_4
   - divide R_4 in half; take the *right* rectangle to be R_5
   - what's the next division?

5. Notice how this "divide and choose" procedure spirals inward. Draw a *smooth* spiral that passes through the successive points where the dividing lines for each rectangle meet the "outside" side of the respective rectangle (the side that's "away from" the spiral of rectangles).

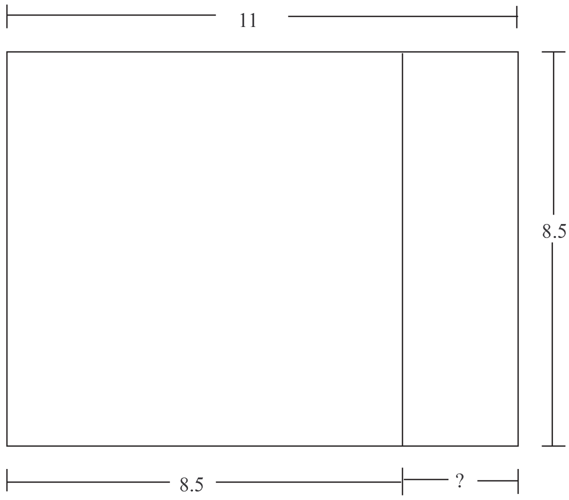
6. What point C does the spiral approach? That is, what's the center of the spiral? Draw the diagonal of R_0 that appears to pass near the center. Do the same for R_1. Do you see why the intersection of these two diagonals must be C? Describe why this is so.

7. Finally, what are the "coordinates" of C—how far from the left and bottom edges, respectively, of R_0 is C? (Knowing that C is the intersection of the R_0 and R_1 diagonals is very useful here.)

8. Have we seen this sort of spiral before?

2.9 The golden spiral

1. Take an 8.5×11 sheet of paper and divide it into two pieces so that the left piece is a square. Is the rectangle on the right similar to the original sheet?



2. We'll now follow a procedure similar to the previous case. How long should a piece of the paper with width = 1 be in order to get a rectangle similar to the original after cutting the original into a square and rectangle?

   Again, think of the length as an unknown G (see Figure 11). Divide the "$G \times 1$" rectangle S_0 into a square on the left and a rectangle S_1 on the right. What value must G have in order for S_1 to be similar to S_0? This number is known as the **Golden Number** and the corresponding rectangle is a **Golden Rectangle**.

3. Use the Pythagorean Theorem to find a crease that you can use to "measure off" the length G. Make a Golden Rectangle S_0 and divide it into a square and rectangle S_1.

4. Now divide S_1 into a square at the bottom and rectangle S_2 at the top. Is S_2 similar to S_1 and, thereby, to S_0? Continue this square-rectangle division procedure in the same sort of spiral manner as before.

5. As before, draw a smooth spiral through the points where the dividing lines meet the outside of the golden rectangles. Use the intersecting diagonals of the two rectangles S_0 and S_1 to find the center. What angle do the two diagonals form?

6. Express the "coordinates" of the center. That is, find the horizontal distance from the left edge of the rectangle S_0 to the center and the vertical distance from the bottom edge to the center.

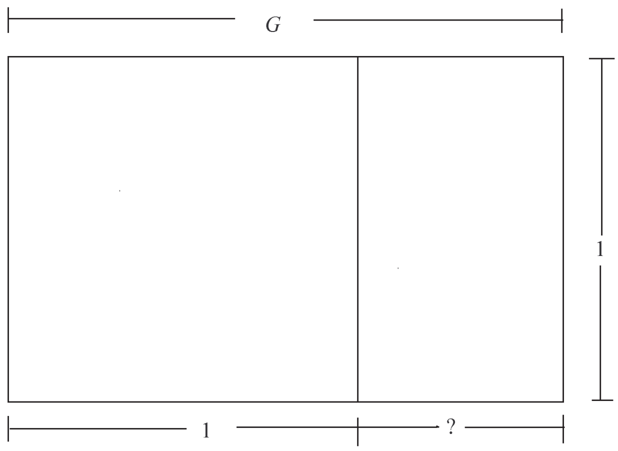


Figure 11: The Golden Rectangle

7. Have we seen this sort of spiral before?

2.10 Rabbits and the golden ratio

You've just gone into the rabbit-breeding business. Being cautious, you start with a single pair of *immature* rabbits. After one month the pair matures to breeding capacity and produces a second pair during the second month. So, at the end of the second month, you have two pairs. During the third month the original pair produces another pair while the second pair spends the time maturing. After three months, you have three pairs, two of which are now mature for breeding. During the fourth month, the two mature pairs each produce a pair leaving a total of five at the end of the month.

Opportunities

1. If the rabbits continue in this fashion, how many pairs are there after five months? Six months? Seven months? Do you see a numerical way of producing the total for the next month? What is it?

2. Call the original number of pairs F_0 so that $F_0 = 1$. After one month the total is $F_1 = 1$. After two months you have $F_2 = 2$. After three months, $F_3 = 3$. So, F_n gives the number of pairs after n months.

Express the pattern for producing the numbers of pairs in terms of the symbol F_n. That is, translate the words that describe the pattern into the symbols that represent the numbers of pairs. These are *Fibonacci's numbers.*[3] Compute the first eleven Fibonacci numbers $F_0, \ldots, F_{10}$.

3. Now form the ratios of consecutive Fibonacci numbers:

$$\frac{F_1}{F_0}, \frac{F_2}{F_1}, \frac{F_3}{F_2}, \ldots, \frac{F_{10}}{F_9}.$$

   Use a calculator to express these numbers in approximate decimal form (say to five decimal places). What seems to happen with to these ratios? Compare these numbers to a decimal approximation of the golden ratio.

4. What's going on here? A clue is to compare the defining expression for the Fibonacci numbers—assuming that you got it right— to the defining equation for the golden ratio

$$x^2 = x + 1 \qquad \text{or} \qquad x^2 = x^1 + x^0.$$

   We'll work on this in class.

5. Create a Fibonacci-like sequence

$$a_0,\ a_1,\ a_2 = a_0 + a_1,\ a_3 = a_1 + a_2, \ldots$$

   by adding two numbers to get the next one. Except, start with

$$a_0 = 1,\ a_1 = 3.$$

   What seems to happen to the ratios of consecutive terms

$$\frac{a_1}{a_0},\ \frac{a_2}{a_1},\ \frac{a_3}{a_2},\ \frac{a_4}{a_3},\ \ldots?$$

   Compare this to the ratios of consecutive Fibonacci numbers. Explain what's going on.

3 Paper-folding

3.1 Folding a letter

Suppose you have a piece of paper that's one unit (feet, meters, whatever) *long.* The width doesn't matter. Hold the paper so that the length is horizontal. Call the left edge L and the right edge R.

[3]Named for the medieval scholar Leonardo of Pisa who was known as Fibonacci.

You want to fold this paper into thirds—like a business letter. How do you know where to make the first crease? You don't. Nevertheless, you make one anyway. Suppose that your crease is one-fourth of the way from L to R. Refer to the distance from L of this "left-crease" as L_1. Now you wonder, what if I place the right edge R into the crease and press the paper down to form a second crease. How far from R is this new crease? Call this length R_1. (See Figure 12 Is R_1 closer to being $1/3$ than is L_1?

Continue this procedure by placing L into the R_1 crease and folding to form a new left-crease whose distance from L is L_2. What is L_2?

Now place R into the L_2 crease and obtain a new right-crease whose distance from R is R_2. What is R_2?

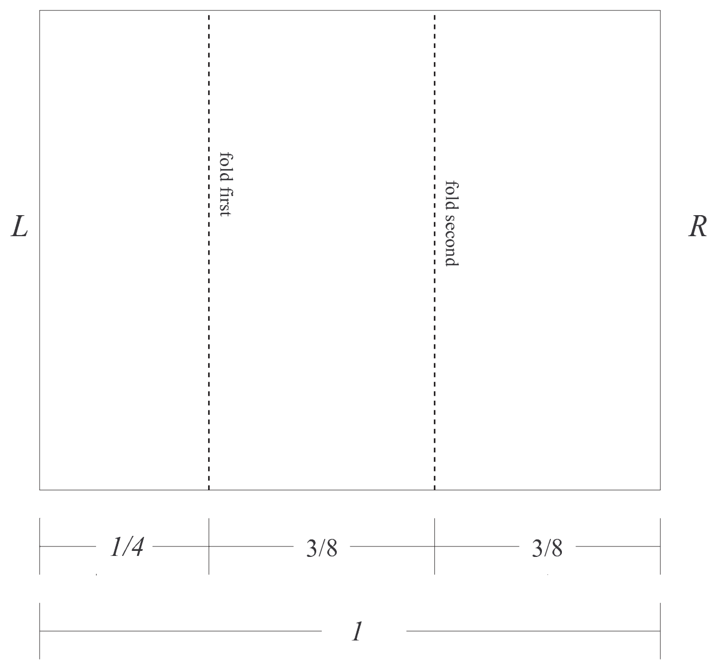


Figure 12: Where's the next crease?

Opportunities

1. Describe the next two steps in this process. What are the lengths L_3 and R_3?

2. Continue folding into the previous crease until you detect a pattern in

the lengths L_n, R_n. Express each of these patterns—that is, describe

$$L_n = \text{an expression in terms of } n$$

$$R_n = \text{an expression in terms of } n.$$

There are different patterns to be found here. Can you detect more than one?

3. Can you tell what happens to these numbers as n gets larger and larger—as you make more and more creases? If not, you might look for a different pattern for the lengths. Now, we'll actually prove that what seems to happen does happen—remember the Fibonacci ratios getting close to the golden ratio.

4. Now, suppose your first fold is some random distance L_1 from the left edge L. (See Figure 13.) Use the above *fold-into-the-previous-crease* procedure to produce subsequent creases whose distances from the respective edges are

$$L_2, L_3, L_4, \ldots$$

$$R_1, R_2, R_3, R_4, \ldots.$$

5. Look for a pattern in these distances—compare the results to the ones that we got before. Express this pattern as

$$L_n = \text{an expression in terms of } n \text{ and } L_1$$

$$R_n = \text{an expression in terms of } n \text{ and } L_1.$$

6. What happens as you continue folding and n gets arbitrarily large?

7. Suppose you fold from the left *twice* before folding from the right *once*. That is the folding procedure goes left-left-right, left-left-right, etc. Do the creases approach a definite value? If so, what is that value?

3.2 To err is human, to measure it is math

Here's another way to look at the letter-folding procedure. Chances are that your first crease isn't exactly right. That is, L_1 is not quite equal to $1/3$. Let's say that you're off by an amount E (for "error"). This means that

$$L_1 = \frac{1}{3} + E.$$

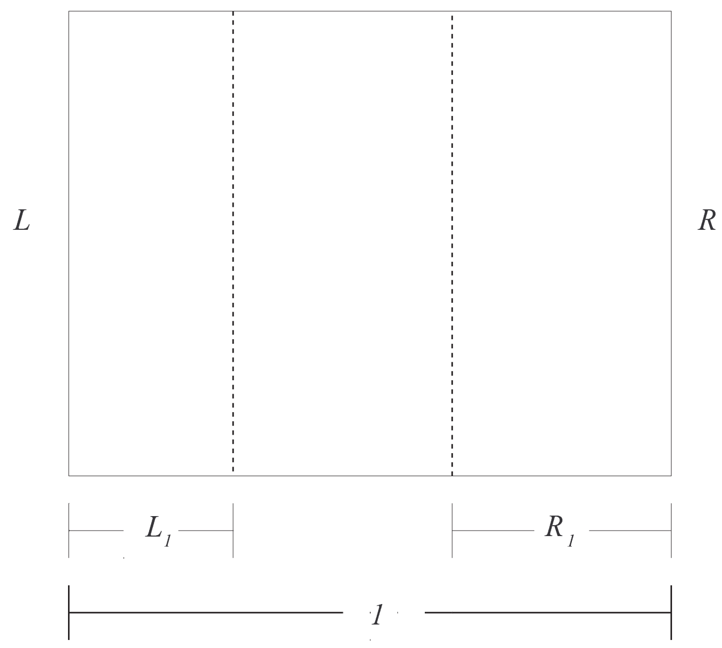


Figure 13: Where's the next crease?

Opportunities

1. With this way of expressing L_1, what's R_1 in terms of the error E? How much in error is R_1?
2. Now find L_2 and R_2 in terms of E. What's the corresponding error for each crease?
3. What does the general pattern tell you? What's happening to the error as you make creases? Does this agree with what you found above?

3.3 Folding a triangle

Take a piece of adding machine tape that's about three feet long. While holding it on the horizontal and starting close to the left end, make an arbitrary crease from the top edge to the bottom. Call the angle that the crease makes with the top edge t_1.

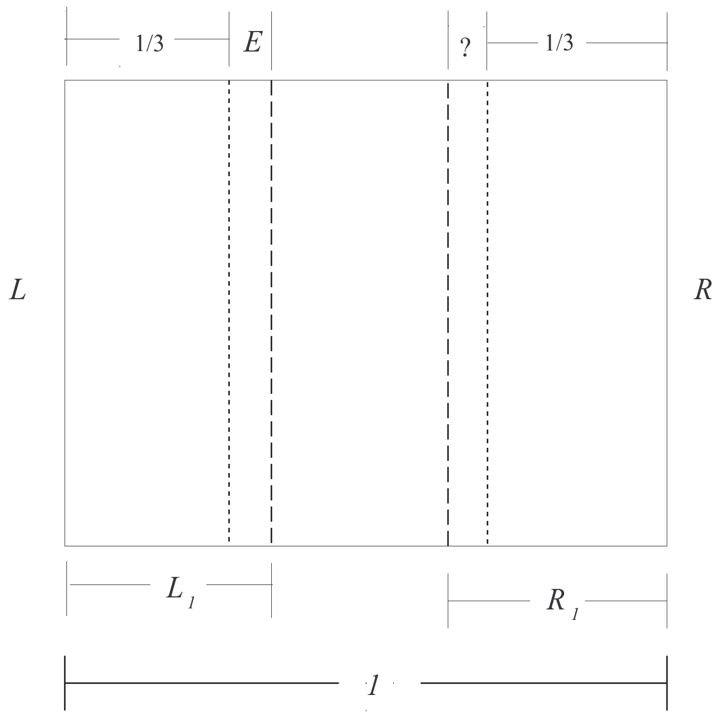


Figure 14: How far off is R_1?

Opportunities

1. This first fold creates two angles at the bottom of the page. The one to the left of the crease is t_1—do you see why? Now fold the angle to the right of the crease *in half*. Refer to this half-angle as b_1. In terms of t_1, express b_1?

2. The new crease makes two angles at the top edge. Fold the one to the right of the crease in half and call the resulting half-angle t_2. Express t_2 in terms of t_1.

3. Continue this procedure to get the half angle b_2 along the bottom and t_3 along the top. Again, express b_2 and t_3 in terms of t_1. Do you detect a pattern? Make a table of angle values compared to number of "top or bottom folds".

4. What seems to happen to the angles? Do they appear to be getting closer to some definite value? How can you be sure? Call the angle that you think they're approaching a. The angle a should be a *specific*

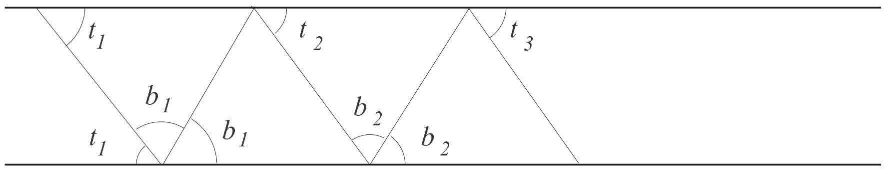


Figure 15: What are the other angles in terms of t_1?

numerical value—the value to which you think the angles on the tape are approaching. As we did in the letter-folding problem, introduce the "error" e relative to a. This means

$$t_1 = a + e.$$

5. As you did before with lengths, express the subsequent angles b_1, t_2, b_2, and t_3 in terms of a and e. What happens? Can you tell that the angles t_2, t_3 are getting closer to a? If not, what do you conclude about your guess of a?

6. Continue folding until you reach the other end of the strip. Use the outcome of folding the tape this way to make an equilateral triangle whose sides are pieces of tape that are 3-4 inches long. To do this you must be able to fold an angle close to one that occurs in the triangle—which is . . . ?

7. Do you notice any similarities between this *angle-folding* procedure and the *length-folding* process of the letter-folding? Briefly discuss them.

3.4 Folding a pentagon

Can we use this sort of folding procedure to make a *regular* pentagon—five equal sides?

Opportunities

1. To make an equilateral triangle we had to get close to an angle of $60°$ or $\pi/3$ radians. What angle must we get close to in order to make a regular pentagon? (See Figure 16.)

2. The first thing to ask is what angle is formed inside the pentagon at a vertex? We want to fold the tape so that the two pieces form the sides of a pentagon. (See Figure 17.) What angle[4] c must the crease make with the top edge in order for the two folded pieces to have the appropriate angle between them?

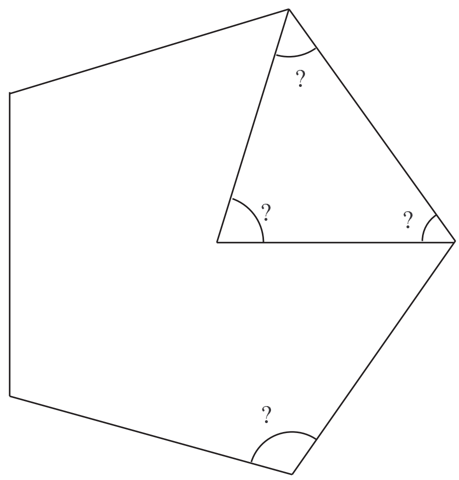


Figure 16: A regular pentagon—what are the angles?

3. So, we want a folding procedure that will produce angles that get close to c. Also, $c/2$ will work—do you see why? We can approach this experimentally—devise a procedure and see what the results are. Get a piece of tape that's about five feet long. Instead of alternating top and bottom when halving the angles to the right of the previous crease, we could halve the bottom angle *twice* and then halve the top angle *twice*. (See Figure 18.)

4. Express the angles b_1, t_2, b_2, and t_3 in terms of the initial angle t_1. Can you make a guess as to what value the angles approach? Look for a pattern in these angles. Try to find a relationship between between the numerators and denominators. Express the general angles t_n and b_n in terms of n. What happens when n gets large?

5. Continue folding until you reach the end of the tape. use the folds that

[4] Again, this should be a specific numerical value.

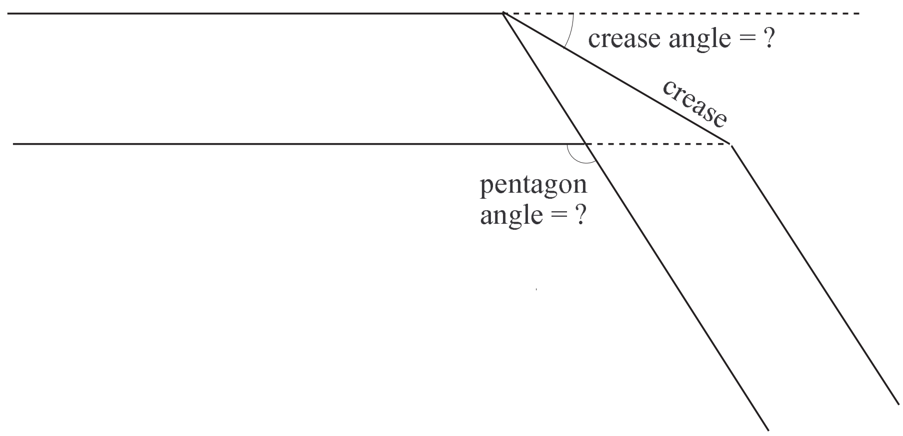


Figure 17: Folding a pentagon

are closest to being the pentagon angle c to create a pentagon. There are several ways of doing this. Experiment.

6. Now, suppose that t_1 happens to be a "perfect" fold—it just happens to be the angle that the subsequent top angles tend towards. This should mean that t_2 is also perfect. Do you see why? In other words,

$$t_2 = t_1.$$

You've already expressed t_2 in terms of t_1. Now, solve the equation $t_2 = t_1$ for t_1.

7. Call this "perfect fold angle" p. Use this value to show that the top angles *really do* approach p. As before, you can represent the first fold angle as

$$t_1 = p + e$$

and express the subsequent fold angles b_1, t_2, b_2, ... in terms of p and e. What happens as you make more and more creases?

3.5 General patterns in angle-folding

The triangle and pentagon procedures give us a way of approximating certain angles. The triangle *algorithm* took an initial angle and then

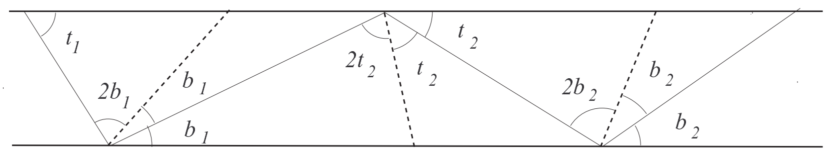


Figure 18: What are the other angles in terms of t_1?

1. divided the "bottom angle" in half (one "up" fold)
2. divided the new top angle in half (one "down" fold)
3. repeat the one up/one down fold process.

We could call this (1,1) folding (Down-1,Up-1). The pentagon procedure involves (2,2) folding. It takes an initial angle and then

1. divides the "bottom angle" into fourths (two "up" folds)
2. divides the new top angle into fourths (two "down" fold)
3. repeats the two up/two down fold process.

What if we made T down-folds on the top angles and B up-folds on the bottom angles?

Opportunities

1. Draw a picture of a piece of tape that illustrates this (T, B) procedure.
2. Again, call the first top-angle t_1. To get the first bottom angle b_1
   (a) fold the angle at the bottom and to the right of the t_1 crease in half
   (b) fold the angle to the right of the crease that you just made in half
   (c) fold the angle to the right of the crease that you just made in half
   (d) repeat this up-folding until you've made B "up-creases"
   (e) call the bottom angle to the right of the last crease b_1—express b_1 in terms of t_1.

To get the second top angle t_2

(a) fold the angle at the top and to the right of the b_1 crease in half

(b) fold the angle to the right of the crease that you just made in half

(c) fold the angle to the right of the crease that you just made in half

(d) repeat this up-folding until you've made T "down-creases"

(e) call the bottom angle to the right of the last crease t_2—express t_2 in terms of t_1.

3. If t_1 happened to be *perfect*, what should t_2 be? Solve for t_1 in the equation that you get. This is your *candidate* for being a perfect fold angle. As before, let's call it p.

4. Why does (T, B) folding produce top angles that are arbitrarily close to p? What happens to the bottom angles?

5. Make a table of the angles that the (T, B) procedure approximates for the following values of T and B:

T	B	angle approximated
1	1	?
1	2	?
2	1	?
2	2	?

A Ideas for a course project

Spirals

- The *cochlea* is a spiral-shaped organ in the inner ear that is crucial in hearing. Stretched across the cochlea is the *basilar membrane* that communicates sound vibrations to nerves and, eventually, the brain. Find out more about this system. Pay special attention to the role that the spiral shape plays in this stage of hearing.

- Make a spiral that's a coiled-up cone and compare it to a spiral made by a coiled rope (cylinder). Discuss why this shape is the one that a snail and nautilus make.

- Investigate spiral spider webs. What kinds of spirals do spiders make? Are there any mathematical reasons why they do this? Can you formulate a hypothesis?

  Sources:

J. Comstock, *The Spider Book*	QL 457.1 C7
J. Crompton, *The Life of the Spider*	QL 452 L3

- Investigate snails or nautiluses in more detail. What biological or physical activity is involved in the construction of their shells?

- **Pythagorean spirals.** These are constructed on the basis of right triangles (similar to the way we made a snail spiral). Here are two that you might explore:

  1. The *non-radial* side of each triangle is the same length.
  2. All three side lengths of each triangle is an integer. You could begin with a (3,4,5) triangle. What's the next triangle?

- **Weather spirals.** Examine a case of meteorological spirals: hurricanes, tornadoes, ocean currents. How do they form? Analyze the kind of spiral that's produced. Is it one that we've studied?

- Why does water spiral down a drain? Can you answer the age-old question of whether the direction of the spiral in the northern hemisphere is opposite to the direction in the southern hemisphere?

Similarity

- Using two sticks and the relationships between each other and the sun Eratosthenes measured the earth's size. Build and describe a model of his technique.

- Investigate and describe some aspect of "shadow casting." How does an object's shadow differ in size and shape from the object itself? How does the relationship between the object and its shadow change as the position of the light source changes? As the kind of light source changes?
- Use the concept of similarity to examine the size of things. How large can a person be? What about a tree? A building?
- **Fractals.** Find out the basics of fractal shapes. What is a fractal? How are they described mathematically? What are some examples?
- An isosceles triangle T_0 with angles $36°, 72°, 72°$ can be cut into two triangles one of which is similar to T_0. Assume that the two equal sides have length 1. Use similarity to find the length of the third side.

  Use T_1 to refer to the triangle that's similar to T_0. Now divide T_1 into two triangles one of which is similar to T_1 and, of course, to T_0. Call that triangle T_2. There are *two different ways* of dividing T_1. Choose the one that gives the "cut-up" version of T_1 the same "handed-ness" as the cut-up version of T_0.

  Continue this procedure: cut T_2 into triangles where one triangle T_3 is similar to T_2 and the cut has the same handed-ness as T_2. Study the spiral structure that results. Draw a spiral that's "fits with" this picture. What kind of spiral is it?

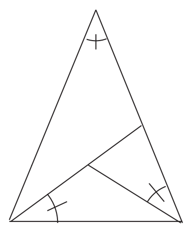


Figure 19: The golden triangle

Paper-folding

- Try to develop a paper-folding procedure for folding a piece of paper into fifths. Describe how you looked at the problem, what you tried, what happened, etc.

- Create your own folding procedure and determine its effect.

- Construct polyhedra by means of paper folding.

  Source: Hilton and Pederson, *Build Your Own Polyhedra*

Symmetry

- One of the most common examples of symmetry (albeit on a microscopic scale) is found in crystals. What is the mathematical description of a crystal? Some examples are snowflakes, diamond, and pencil lead (graphite). Study an example and describe how a mathematical understanding of the crystal is valuable for understanding its properties.

- **God's favorite number.** Find examples of 5-fold symmetry in nature. Is there some physical, biological, etc. connection between these cases? Speculate on what physical, biological, etc. purposes this common form of symmetry plays.

- Explore and describe the relationships between the lengths of various pieces of the pentagram (five-point star).

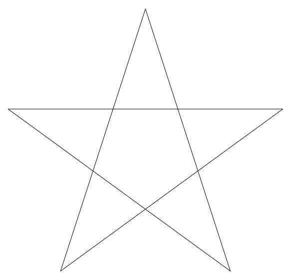


Figure 20: The pentagram

- Build and discuss a "polyhedral kaleidescope." This is made by joining several mirrors together in such a way that the "mirror images" produce the appearance of a whole regular polyhedron.
- Create tie-dye patterns that exhibit various types of symmetry. E.g.,
  - spiral
  - reflective
  - rotational
  - ??

  Describe how the patterns are made and what the underlying math is.
- **Symmetry as evolutionary advantage.** Investigate a specific case in which an animal or plant's symmetry (or lack thereof) provides a competitive advantage or disadvantage in the struggle for survival. (Source: Thompson, *The Growth of Form.*)
- What shapes do snowflakes form? What kinds of symmetry do they exhibit? Are there mathematical reasons for the shapes that they have?

Tiling and polyhedra

- A familiar sort of tiling is a **frieze** pattern. Basically, this is a tiling of a strip. Wallpaper borders are a typical example. There are only seven different frieze patterns—in the sense of having different symmetries. Investigate and illustrate these patterns and discuss how to see that there are just seven.

  Source: Boles and Newman, *The Surface Plane.*
- Examine the tiling on a turtle's shell. What shape are the tiles? How do they fit together? What significance does the shell's curved shape play? How is such a shell made and what are the mathematical considerations here?
- Describe and demonstrate how to build a dodecahedron by erecting a "roof" on each face of a cube. How do twelve faces appear where there were just six to start with?
- Investigate the structure of the soccer ball as a polyhedron. What are its faces? $V, E, F=$?? What are its symmetries? What relationship does it have to the dodecahedron?

Art

- Examine how architects/artists have used the golden ratio. Perhaps consider a specific building or art work.
- Find and consider an example/examples of art (visual, musical, performing). Is there symmetry present? Similarity? Spiral structure? Something else? "Op-art" (visual illusion) is particularly good. How does the image affect/trick the eye?

B Resources and references

Reference materials

Background geometry and algebra

Smart, *Introductory Geometry: An Informal Approach*

Garner and Nunley, *Geometry; An Intuitive Approach*

Spirals

Gardner, *The Ambidextrous Universe: Mirror Asymmetry and Time-reversed Worlds*

Ghyka, *The Geometry of Art and Life*

Huntley, *The Divine Proportion: A Study of Mathematical Beauty*

Linn, *The Golden Mean: Mathematics and the Fine Arts*

Thompson, *On Growth and Form*

Paper-folding and polyhedra

Hilton and Pederson, *Build Your Own Polyhedra*

Ody, *Paper-folding and Paper Sculpture*

Pearce and Pearce, *A Polyhedra Primer*

Pugh, *Polyhedra: A visual approach*

Wenninger, *Polyhedron Models*

Symmetry

Belmonte, "How the body tells left from right," *Scientific American*, June 1999

P. Gerdes, *Geometry from Africa : Mathematical and Educational Explorations*

Holden, *Shapes, Space, and Symmetry*

Kepes, *Module, Proportion, Symmetry, Rhythm*

Kinsey and Moore, *Symmetry, Shape, and Space: An Introduction to Mathematics Through Geometry*

Martin, *Transformation Geometry: An Introduction to Symmetry*

Rosen, *Symmetry Discovered: Concepts and Applications in Nature and Science*

Ranucci, *Creating Escher-type Drawings*

Senechal and Fleck, eds., *Patterns of Symmetry*

Art

Abas, *Symmetries of Islamic Geometrical Patterns*

Critchlow, *Islamic Patterns: An Analytical and Cosmological Approach*

Escher, *The World of M.C. Escher*

Videos and guidebooks

These items might help with some of the math background material. They are available in the media section of the library.

Similarity
Video Cassette 6482

The Story of π
Video Cassette 6226

The Theorem of Pythagoras
Video Cassette 6225