Symmetry through the Eyes of a Chemist

Second Edition

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behavior. In solid/gas reactions, for example, crystal polarity may be a source of considerable anisotropy.

There are also important physical properties characterizing polar crystals, such as pyroelectricity and piezoelectricity and others [2-34]. The primitive cell of a pyroelectric crystal possesses a dipole moment. The separation of the centers of the positive and negative charges changes upon heating. In this process the two charges migrate to the two ends of the polar axis. Piezoelectricity is the separation of the positive and negative charges upon expansion/compression of the crystal. Both pyroelectricity and piezoelectricity have practical uses.

2.7 CHIRALITY

There are many objects, both animate and inanimate, which have no symmetry planes but which occur in pairs related by a symmetry plane and whose mirror images cannot be superposed. Figure 2-45 shows a building decoration, a detail from Bach’s The Art of the Fugue, a pair of molecules, and a pair of crystals. The simplest chiral molecules are those in which a carbon atom is surrounded by four different ligands—atoms or groups of atoms—at the vertices of a tetrahedron. All the naturally occurring amino acids are chiral, except glycine.
W. H. Thomson, Lord Kelvin, wrote [2-35]: "I call any geometrical figure or group of points 'chiral,' and say it has chirality, if its image in a plane mirror, ideally realized, cannot be brought into coincidence with itself." He called forms of the same sense homochiral and forms of the opposite sense heterochiral. The most common example of a heterochiral form is hands. Indeed, the word chirality itself comes from the Greek word for hand. Figures 2-46 and 2-47 show some heterochiral and homochiral pairs of hands.

A chiral object and its mirror image are enantiomorphous, and they are

Figure 2-46. Heterochiral pairs of hands: (a) Tombstone in the Jewish cemetery, Prague; photograph by the authors; (b) Albrecht Dürer's Praying Hands on the cover of the German magazine Der Spiegel, June 15, 1992; reproduced by permission; (c) Buddha in Tokyo; photograph by the authors; (d) United Nations stamp. (Continued on next page)

Figure 2-47. Homochiral pairs of hands: (a) Cover of the German magazine Der Spiegel, May 18, 1992; reproduced by permission; (b) U.S. stamp; (c) logo with SOS distress sign at a Swiss railway station; photograph by the authors.
each other’s enantiomorphs. Louis Pasteur (Figure 2-48) first suggested that molecules can be chiral. In his famous experiment in 1848, he recrystallized a salt of tartaric acid and obtained two kinds of small crystals which were mirror images of each other, as shown by Pasteur’s models in Figure 2-49. The two kinds of crystals had the same chemical composition but differed in their optical activity. One was levo-active (L), and the other was dextro-active (D). Since the true absolute configuration of molecules could not be determined at the time, an arbitrary convention was applied, which, luckily, proved to coincide with reality. If a molecule or a crystal is chiral, it is necessarily optically active. The converse is, however, not true. There are, in fact nonenantiomorphous symmetry classes of crystals which may exhibit optical activity.

Whyte [2-37] extended the definition of chirality as follows: “Three-dimensional forms (point arrangements, structures, displacements, and other processes) which possess non-superposable mirror images are called ‘chiral’.” A chiral process consists of successive states, all of which are chiral. The two main classes of chiral forms are screws and skews. Screws may be conical or cylindrical and are ordered with respect to a line. Examples of the latter are the left-handed and right-handed helices in Figure 2-50. The skews, on the other hand, are ordered around their center. Examples are chiral molecules having point-group symmetry.

From the point of view of molecules, or crystals, left and right are intrinsically equivalent. An interesting overview of the left/right problem in science has been given by Gardner [2-39]. Distinguishing between left and right has also considerable social, political, and psychological connotations. For example, left-handedness in children is viewed with varying degrees of tolerance in different parts of the world. Figure 2-51a shows a classroom at the University of Connecticut with different (homochiral and heterochiral) chairs.

**Figure 2-48.** Louis Pasteur’s bust in front of the Pasteur Institute, Paris. Photograph by the authors.

**Figure 2-49.** Pasteur’s models of enantiomeric crystals in the Pasteur Institute, Paris. Photographs by the authors.

**Figure 2-50.** Left-handed and right-handed helix decorations from Zagorsk, Russia [2-38]. Photograph by the authors.
Figure 2-51. Classrooms with heterochiral and homochiral chairs: (a) Chairs for both the right-handed and left-handed students; (b) Older chairs, all for right-handed students only. Photographs by the authors.

to accommodate both the right-handed and the left-handed students. Older classrooms at the same university have chairs for the right-handed only (Figure 2-51b).

2.7.1 Asymmetry and Dissymmetry

Symmetry operations of the first kind and of the second kind are sometimes distinguished in the literature (cf. Ref. [2-40]). Operations of the first kind are sometimes also called even-numbered operations. For example, the identity operation is equivalent to two consecutive reflections from a symmetry plane. It is an even-numbered operation, an operation of the first kind. Simple rotations are also operations of the first kind. Mirror rotation leads to figures consisting of right-handed and left-handed components and therefore is an operation of the second kind. Simple reflection is also an operation of the second kind as it may be considered as a mirror rotation about a onefold axis. A simple reflection is related to the existence of two enantiomorphic components in a figure. Figure 2-52 illustrates these distinctions by a series of simple sketches after Shubnikov [2-40]. In accordance with the above description, chirality is sometimes defined as the absence of symmetry elements of the second kind.

Pasteur used "dissymmetry" for the first time as he designated the absence of elements of symmetry of the second kind in a figure. Accordingly, dissymmetry did not exclude elements of symmetry of the first kind. Pierre Curie suggested an even broader application of this term. He called a crystal dissymmetric in the case of the absence of those elements of symmetry upon which depends the existence of one or another physical property in that crystal. In Pierre Curie's original words [2-41], "Dissymmetry creates the phenomenon." Namely a phenomenon exists and is observable due to dissymmetry, i.e., due to the absence of some symmetry elements from the system. Finally, Shubnikov [2-40] called dissymmetry the falling out of one or another element of symmetry from a given group. He argued that to speak of the absence of elements of symmetry makes sense only when these symmetry elements are present in some other structures.

Thus, from the point of view of chirality any asymmetric figure is chiral, but asymmetry is not a necessary condition for chirality. All dissymmetric figures are also chiral if dissymmetry means the absence of symmetry elements of the second kind. In this sense, dissymmetry is synonymous with chirality.

An assembly of molecules may be achiral for one of two reasons. Either all the molecules present are achiral or the two kinds of enantiomorphs are present in equal amounts. Chemical reactions between achiral molecules lead to achiral products. Either all product molecules will be achiral or the two kinds of chiral molecules will be produced in equal amounts. Chiral crystals may sometimes be obtained from achiral solutions. When this happens, the two enantiomorphs will be obtained in (roughly) equal numbers, as was observed by Pasteur. Quartz crystals are an inorganic example of chirality (Figure 2-45d). Roughly equal numbers of left-handed and right-handed crystals are obtained from the achiral silica melt.
Incidentally, Pierre Curie's teachings on symmetry are probably not so widely known as they should be, considering their fundamental and general importance. The fact that his works on symmetry were characterized by extreme brevity may have contributed to this. Marie Curie and Aleksei V. Shubnikov [2-42, 2-43] have considerably facilitated the dissemination of Curie's teachings. Our discussion also relies on their works. There is also a critical and fascinating discussion of Pierre Curie's symmetry teachings by Stewart and Golubitsky [2-44].

Pierre Curie's above-quoted statement concerning the role of dissymmetry in "creating" a phenomenon is part of a broader formulation. It states that in every phenomenon there may be elements of symmetry compatible with, though not required by, its existence. What is necessary is that certain elements of symmetry shall not exist. In other words, it is the absence of certain symmetry elements that is a necessary condition for the phenomenon to exist.

Another important statement of Pierre Curie's is that when several phenomena are superposed in the same system, the dissymmetries are added together. As a result, only those symmetry elements which were common to each phenomenon will be characteristic of the system.

Finally, concerning the symmetry relationships of causes and effects, Marie Curie [2-42] formulated the following principles from Pierre Curie's teachings. (1) "When certain causes produce certain effects, the elements of symmetry in the causes ought to reappear in the effects produced"; (2) "When certain effects reveal a certain dissymmetry, this dissymmetry should be apparent in the causes which have given them birth"; however, (3) "The converse of these two statements does not hold . . . [and] the effects produced can be more symmetrical than their causes."

### 2.7.2 Relevance to Origin of Life

The situation with respect to living organisms is unique. Living organisms contain a large number of chiral constituents, but only L-amino acids are present in proteins and only D-nucleotides are present in nucleic acids. This happens in spite of the fact that the energy of both enantiomorphs is equal and their formation has equal probability in an achiral environment. However, only one of the two occurs in nature, and the particular enantiomorphs involved in life processes are the same in humans, animals, plants, and microorganisms. The origin of this phenomenon is a great puzzle which, according to Prelog [2-45], may be regarded as a problem of molecular theology.

This problem has long fascinated those interested in the molecular basis of the origin of life (e.g., Refs. [2-46], [2-47]). There are in fact two questions. One is why do all the amino acids in a protein have the same L-configuration, or why do all the components of a nucleic acid, that is, all its nucleotides, have the same D-configuration? The other question, the more intriguing one, is why does that particular configuration happen to be L for the amino acids and why does it happen to be D for nucleotides in all living organisms? This second question seems to be impossible to answer satisfactorily at the present time.

According to Prelog [2-45], a possible explanation is that the creation of living matter was an extremely improbable event, which occurred only once. We may then suppose that if there are living forms similar to ours on a distant planet, their molecular structures may be the mirror images of the corresponding molecular structures on the earth. We know of no structural reason at the molecular level for living organisms to prefer one type of chirality. (There may be reasons at the atomic nuclear level. The violation of parity at the nuclear level has already been referred to in Chapter 1.) Of course, once the selection is made, the consequences of this selection must be examined in relation to the first question. The fact remains, however, that chirality is intimately associated with life. This means that at least dissymmetry and possibly asymmetry are basic characteristics of living matter.

Although Pasteur believed that there is a sharp gap between vital and nonliving processes, he attributed the asymmetry of living matter to the asymmetry of the structure of the universe and not to a vital force. Pasteur himself wrote that he was inclined to think that life, as it appears to us, must be a product of the dissymmetry of the universe (see Ref. [2-48]).

Concerning the first question, Orgel [2-46] suggests that we compare the structure of DNA to a spiral staircase. The regular DNA right-handed double helix is composed of D-nucleotides. On the other hand, if a DNA double helix were synthesized from L-nucleotides, it would be left-handed. These two helices can be visualized as right-handed and left-handed spiral staircases, respectively. Both structures can perform useful functions. A DNA double helix containing both D- and L-nucleotides, however, could not form a truly helical structure at all since its handedness would be changing. Just consider the analogous spiral staircase that Orgel suggested as shown in Figure 2-53.

If each component of a complex system is replaced by its mirror image, the mirror image of the original system is obtained. However, if only some components of the complex system are replaced by their mirror images, a chaotic system emerges. Chemical systems that are perfect mirror images of each other behave identically, whereas systems in which only some, but not all, components have been replaced by their mirror images have quite different chemical properties. If, for example, a naturally occurring enzyme made up of L-amino acids synthesizes a D-nucleotide, then the corresponding artificial enzyme obtained from D-amino acids would synthesize the L-nucleotide. On the other hand, a corresponding polypeptide containing both D- and L-amino acids would probably lack the enzymic activity.

Recently, the first enzymatically active D-protein has been synthesized...
There are many potential applications, both therapeutic and nontherapeutic, that may open up with such progress [2-50]. It has been known for some time that the two enantiomers of drugs and pesticides may have vastly different responses in a living organism. Natural products extracted from plants and animals are enantiomerically pure while the synthesized ones are obtained in a 1:1 ratio of the enantiomers. In some cases, the twin of the one exerting the beneficial action is harmless. In other cases, however, the drug molecule has an "evil twin" [2-51]. A tragic example was the thalidomide case, in Europe, in which the right-handed molecule was a sedative and the left-handed one caused birth defects. Other examples include one enantiomer of ethambutol fighting tuberculosis with its evil twin causing blindness, and one enantiomer of naproxen reducing arthritic inflammation with its evil twin poisoning the liver. Bitter and sweet asparagine are represented by structural formulas in Figure 2-54.

Ibuprofen is a lucky case in which the twin of the enantiomer that provides the curing is converted to the beneficial version by the body.

Even when the twin is harmless, it represents waste and a potential pollutant. Thus, a lot of efforts are directed toward producing enantiomerically pure drugs and pesticides. The techniques of asymmetric synthesis (see, e.g., Refs. [2-53] and [2-54]), based on the strategy of employing chiral catalysts (see, e.g., Ref. [2-55]), are used to this end. One of the fascinating possibilities is to produce sweets from chiral sugars of the enantiomer that would not be capable of contributing to obesity yet would retain the taste of the other enantiomer. Chiral separation and purity is an increasingly important question. Worldwide sales of enantiopure drugs topped $35 billion in 1993 and are expected to reach about $40 billion in 1997 [2-56]. There is a rapidly growing literature on the subject, with even special journals exclusively dedicated to this topic. Production of enantiomerically pure substances has also become a topic in investment reports and the daily press.

2.7.3 La coupe du roi

Among the many chemical processes in which chirality/achirality relationships may be important are the fragmentation of some molecules and the reverse process of the association of molecular fragments. Such fragmentation and association can be considered generally and not just for molecules. The usual cases are those in which an achiral object is bisected into achiral or heterochiral halves. On the other hand, if an achiral object can be bisected into two homochiral halves, it cannot be bisected into two heterochiral ones. A relatively simple case is the tessellation of planar achiral figures into achiral, heterochiral, and homochiral segments. Some examples are shown in Figure 2-55. For a detailed discussion, see Ref. [2-7].

Anet et al. [2-57] have cited a French parlor trick called la coupe du roi—or the royal section—in which an apple is bisected into two homochiral halves, as shown in Figure 2-56. An apple can be easily bisected into two achiral halves. On the other hand, it is impossible to bisect an apple into two heterochiral halves. Two heterochiral halves, however, can be obtained from two apples, both cut into two homochiral halves in the opposite sense (see Figure 2-56).
Figure 2-56. The French parlor trick *la coupe du roi*, after Anet et al. [2-57]. An apple can be cut into two homochiral halves in two ways which are enantiomorphous to each other. An apple cannot be cut into two heterochiral halves. Two heterochiral halves originating from two different apples cannot be combined into one apple.

In *la coupe du roi*, two vertical half cuts are made through the apple—one from the top to the equator, and another, perpendicularly, from the bottom to the equator. In addition, two nonadjacent quarter cuts are made along the equator. If all this is properly done, the apple should separate into two homochiral halves as seen in Figure 2-56.

The first chemical analog of *la coupe du roi* was demonstrated by Cinquini et al. [2-58] by bisecting the achiral molecule *cis*-3,7-dimethyl-1,5-cyclooctanedione into homochiral halves, viz., 2-methyl-1,4-butanediol. The reaction sequence is depicted in Figure 2-57 after Cinquini et al. [2-58], who painstakingly documented the analogy with the pomaceous model. Only examples of the reverse *coupe du roi* had been known prior to the work of Cinquini et al. Thus, Anet et al. [2-57] had reported the synthesis of chiral 4-(bromo-methyl)-6-(mercaptopmethyl)[2.2]metacyclophane. They then showed that two homochiral molecules can be combined to form an achiral dimer as shown in and illustrated by Figure 2-58.
La coupe du roi and the reaction sequence transforming cis-3,7-dimethyl-1,5-cyclooctanone into 2-methyl-1,4-butanediol. After Cinquini et al. [2-58]. Used by permission. Copyright (1988) American Chemical Society.


**2.8 POLYHEDRA**

“A convex polyhedron is said to be regular if its faces are regular and equal, while its vertices are all surrounded alike” [2 59]. A polyhedron is convex if every dihedral angle is less than 180°. The dihedral angle is the angle formed by two polygons joined along a common edge.

There are only five regular convex polyhedra, a very small number indeed. The regular convex polyhedra are called *Platonic solids* because they constituted an important part of Plato’s natural philosophy. They are the tetrahedron, cube (hexahedron), octahedron, dodecahedron, and icosahedron. The faces are regular polygons, either regular triangles, regular pentagons, or squares.

A regular polygon has equal interior angles and equal sides. Figure 2-59 presents a regular triangle, a regular quadrangle (i.e., a square), a regular pentagon, and so on. The circle is obtained in the limit as the number of sides approaches infinity. The regular polygons have an \( n \)-fold rotational symmetry.
axis perpendicular to their plane and going through their midpoint. Here \( n \) is 1, 2, 3, \ldots up to infinity for the circle.

The five regular polyhedra are shown in Figure 2-60. Their characteristic parameters are given in Table 2-3. Figure 2-61 reproduces an East German stamp with Euler and his equation, \( V - E + F = 2 \), where \( V \), \( E \), and \( F \) are the number of vertices, edges, and faces. The equation is valid for polyhedra having any kind of polygonal faces. According to Weyl [2-9], the existence of the tetrahedron, cube, and octahedron is a fairly trivial geometric fact. On the other hand, he considered the discovery of the regular dodecahedron and the regular icosahedron "one of the most beautiful and singular discoveries made in the whole history of mathematics." However, to ask who first constructed the regular polyhedra is, according to Coxeter [2-59], like asking who first used fire.

Many primitive organisms have the shape of the pentagonal dodecahedron. As will be seen later, it is not possible to have crystal structures having this symmetry. Belov [2-60] suggested that the pentagonal symmetry of primitive organisms represents their defense against crystallization. Several radiolarians of different shapes from Häckel's book [2-13] are shown in Figure 2-62. Artistic representations of regular polyhedra are shown in Figure 2-63.

Figure 2-64 shows Kepler and his planetary model based on the regular solids [2-61]. According to this model, the greatest distance of one planet from the sun stands in a fixed ratio to the least distance of the next outer planet from the sun. There are five ratios describing the distances of the six planets that were known to Kepler. A regular solid can be interposed between two adjacent planets so that the inner planet, when at its greatest distance from the sun, lays

<table>
<thead>
<tr>
<th>Name</th>
<th>Polygon</th>
<th>Number of faces</th>
<th>Vertex figure</th>
<th>Number of vertices</th>
<th>Number of edges</th>
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<td>Icosahedron</td>
<td>3</td>
<td>20</td>
<td>5</td>
<td>12</td>
<td>30</td>
</tr>
</tbody>
</table>
on the inscribed sphere of the solid, while the outer planet, when at its least distance, lays on the circumscribed sphere.

Arthur Koestler in *The Sleepwalkers* [2-62] called this planetary model “a false inspiration, a supreme hoax of the Socratic daimon, . . .”. However, the planetary model, which is also a densest packing model, probably represents Kepler’s best attempt at attaining a unified view of his work both in astronomy and in what we call today crystallography.

There are excellent monographs on regular figures, two of which are especially noteworthy [2-59, 2-63]. The Platonic solids have very high symmetries and one especially important common characteristic. None of the rotational symmetry axes of the regular polyhedra is unique, but each axis is associated with several axes equivalent to itself. The five regular solids can be classified into three symmetry classes:

Figure 2-62. Radiolarians from Hückel’s book [2-13].

Figure 2-63. Artistic representations of regular polyhedra: (a) Sculpture in the garden of Tel Aviv University; photograph by the authors; (b) pentagonal dodecahedron by Horst Janssen, ChrisStaL-KneChi (crystal slave); reproduced by permission; (c) Leonardo da Vinci’s dodecahedron drawn for Luca Pacioli’s De Divina Proporzione; (d) sculpture by Victor Vasarely in Pécs, Hungary; photograph by the authors.
It is equivalent to describe the symmetry class of the tetrahedron as $3/2\cdot m$ or $3/4$. The skew line between two axes means that they are not orthogonal. The symbol $3/2\cdot m$ denotes a threefold axis and a twofold axis which are not perpendicular and a symmetry plane which includes these axes. These three symmetry elements are indicated in Figure 2-65. The symmetry class $3/2\cdot m$ is equivalent to a combination of a threefold axis and a fourfold mirror-rotation axis. In both cases the threefold axes connect one of the vertices of the tetrahedron with the midpoint of the opposite face. The fourfold mirror-rotation axes coincide with the twofold axes. The presence of the fourfold mirror-rotation axis is easily seen if the tetrahedron is rotated by a quarter rotation about a twofold axis and is then reflected by a symmetry plane perpendicular to this axis. The symmetry operations chosen as basic will then generate the remaining symmetry elements. Thus, the two descriptions are equivalent.

Characteristic symmetry elements of the cube are shown in Figure 2-65. Three different symmetry planes go through the center of the cube parallel to its faces. Furthermore, six symmetry planes connect the opposite edges and also diagonally bisect the faces. The fourfold rotation axes connect the midpoints of opposite faces. The sixfold mirror-rotation axes coincide with threefold rotation axes. They connect opposite vertices and are located along the body diagonals. The symbol $6/4$ does not directly indicate the symmetry planes connecting the midpoints of opposite edges, the twofold rotation axes, or the center of symmetry. These latter elements are generated by the others. The presence of a center of symmetry is well seen by the fact that each face and edge of the cube has its parallel counterpart. The tetrahedron, on the other hand, has no center of symmetry.

The octahedron is in the same symmetry class as the cube. The antiparallel character of the octahedron faces is especially conspicuous. As seen in Figure 2-65, the fourfold symmetry axes go through the vertices, the threefold axes go through the face midpoints, and the twofold axes go through the edge midpoints.

The pentagonal dodecahedron and the icosahedron are in the same symmetry class. The fivefold, threefold, and twofold rotation axes intersect the midpoints of faces, the vertices, and the edges of the dodecahedron, respec-
ally (Figure 2-65). On the other hand, the corresponding axes intersect the vertices and the midpoints of faces and edges of the icosahedron (Figure 2-65).

Consequently, the five regular polyhedra exhibit a dual relationship as regards their faces and vertex figures. The tetrahedron is self-dual (Table 2-3).

If the definition of regular polyhedra is not restricted to convex figures, their number rises from five to nine. The additional four are depicted in Figure 2-66 (for more information, see, e.g., Refs. [2-59] and [2-63]-[2-66]). They are called by the common name of regular star polyhedra. One of them, viz., the great stellated dodecahedron, is illustrated by the decoration at the top of the Sacristy of St. Peter's Basilica in Vatican City, and another, the small stellated dodecahedron, by an ordinary lamp in Figure 2-67.

The sphere deserves special mention. It is one of the simplest possible figures and, accordingly, one with high and complicated symmetry. It has an infinite number of rotation axes with infinite order. All of them coincide with body diagonals going through the midpoint of the sphere. The midpoint, which is also a singular point, is the center of symmetry of the sphere. The following symmetry elements may be chosen as basic ones: two infinite order rotation axes which are not perpendicular plus one symmetry plane. Therefore, the symmetry class of the sphere is $\infty/\infty/m$. Concerning the symmetry of the sphere, Kepes [2-67] quotes Copernicus:

> The spherical is the form of all forms most perfect, having need of no articulation; and the spherical is the form of greatest volumetric capacity, best able to contain and circumscribe all else; and all the separated parts of the world—I mean the sun, the moon, and the stars—are observed to have spherical form; and all things tend to limit themselves under this form—as appears in drops of water and other liquids whenever of themselves they tend to limit themselves.

So no one may doubt that the spherical is the form of the world, the divine body.

Artistic appearances of spheres are shown in Figure 2-68.

In addition to the regular polyhedra, there are various families of polyhedra with decreased degrees of regularity [2-59, 2-63-2-66]. The so-called semiregular or Archimedean polyhedra are similar to the Platonic polyhedra in that all their faces are regular and all their vertices are congruent. However, the polygons of their faces are not all of the same kind. The thirteen semiregular polyhedra are listed in Table 2-4, and some of them are also shown in Figure 2-69. Table 2-4 also enumerates their rotation axes.

The simplest semiregular polyhedra are obtained by symmetrically shav-
Simple and Combined Symmetries

Figure 2-68. Artistic expressions of the sphere: (a) In front of the World Trade Center, New York; (b) sculpture by J.-B. Carpeaux in Paris. Photographs by the authors.

Table 2-4. The Thirteen Semiregular Polyhedra

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<tr>
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<td>15</td>
<td>10</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>12</td>
<td>Rhombicosidodecahedron</td>
<td>62</td>
<td>60</td>
<td>120</td>
<td>15</td>
<td>10</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>13</td>
<td>Snub dodecahedron</td>
<td>92</td>
<td>60</td>
<td>150</td>
<td>15</td>
<td>10</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

*a*Truncated regular polyhedron.

*b*Quasiregular polyhedron.

Figure 2-69. Some of the semiregular polyhedra: the so-called truncated regular polyhedra and quasiregular polyhedra.

...ing off the corners of the regular solids. They are the truncated regular polyhedra and are marked with a superscript $a$ in Table 2-4. One of them is the truncated icosahedron, the shape of the buckminsterfullerene molecule. Two semiregular polyhedra are classified as so-called quasiregular polyhedra. They have two kinds of faces, and each face of one kind is entirely surrounded by faces of the other kind. They are marked with a superscript $b$ in Table 2-4. All these seven semiregular polyhedra are shown in Figure 2-69. The remaining six semiregular polyhedra may be derived from the other semiregular polyhedra. The structures of zeolites, aluminosilicates, are rich in polyhedral shapes, including the channels and cavities they form (see, e.g., Ref. [2-68]). One of the most common zeolites is sodalite, $Na_6[Al_6Si_6O_{24}] \cdot 2NaCl$, whose name refers to its sodium content. The sodalite unit itself is represented by a truncated octahedron in Figure 2-70a, where the line drawing ignores the oxygen atoms and each line represents $T \ldots T$ ($T = Al, Si$). The three remaining models of Figure 2-70 (b, c, and d) represent different modes of linkages between the sodalite units. It is especially interesting to see the different cavities formed by different modes of linkage [2-69]. Some other examples of semiregular polyhedra are shown in Figure 2-71.

The prisms and antiprisms are also important polyhedron families. A prism has two congruent and parallel faces, and they are joined by a set of parallelograms. An antiprism also has two congruent and parallel faces, but they are joined by a set of triangles. There is an infinite number of prisms and antiprisms, and some of them are shown in Figure 2-72. A prism or an antiprism is semiregular if all its faces are regular polygons. A cube can be considered a square prism, and an octahedron can be considered a triangular antiprism.

There are additional polyhedra which are important in discussing molecular geometries and crystal structures.
Figure 2-70. Zeolite structures: the shape and various modes of linkage of the sodalite units, after Beagley and Titiloye [2-69]. Reproduced by permission. The line drawings ignore the oxygen atoms and represent T . . . T (T = Al, Si); (a) Sodalite unit; (b) and (c) sodalite units linked through double 4-rings; (d) Sodalite units linked through double 6-rings. In this model double lines represent T . . . T.

Figure 2-71. Examples of semiregular polyhedra: (a) Truncated octahedron in a Tel Aviv playground; (b) truncated icosahedron as a lamp in an Italian home, Bologna; (c) cuboctahedron as a top decoration of a garden lantern in Kyoto, Japan. Photographs by the authors.

REFERENCES


Chapter 2


Simple and Combined Symmetries