Fearful Symmetry

Is God a Geometer?

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Geometer God

> One could perhaps describe the situation by saying that God is a mathematician of a very high order, and he used very advanced mathematics in constructing the universe.
>
> Paul Dirac, *The Evolution of the Physicist's Picture of Nature*

'God Ever Geometrizes'

So (it is said) the Greek philospher Plato stated his belief that the physical universe is governed by laws that can be expressed in the language of mathematics. It isn't necessary to subscribe to the existence of a personal deity to observe that there are patterns and regularities in the world that we inhabit, and to wonder why; but Plato's statement is economical, pointed, and strikes at the heart of the matter. Nature's patterns are mathematical.

Numerous scientists throughout the ages have held similar beliefs. In 1956 Paul Dirac, one of the founders of quantum mechanics, visited the University of Moscow, and was asked to write an inscription on a blackboard, to be preserved for posterity. It was an honour reserved for only the very greatest visitors. Dirac knew this, and selected his personal credo: 'A physical law must possess mathematical beauty.'

Nearly all of today's science is founded on mathematics; indeed the state of maturity of a science is often judged by how mathematical it has become. Even biology, traditionally one of the least mathematical of the true sciences, has become much more so as it delves ever more deeply into the information-processing structures that surround the DNA molecule. Mathematics in its purest form – logic – lies at the heart of genetics, evolution, what it means to be alive, to be human.

Scientists use mathematics to build mental universes. They write down mathematical descriptions – models – that capture essential fragments of how they think the world behaves. Then they analyse their consequences. This is called 'theory'. They test their theories against observations: this is called 'experiment'. Depending on the result, they may modify the mathematical model and repeat the cycle until theory and experiment agree. Not that it's really that simple; but that's the general gist of it, the essence of the scientific method. The real thing, with its emotional commitment, priority disputes, and Nobel prize political infighting, also has sociological aspects. Those arise because science has to be performed by *people*, and people suffer from all sorts of distractions. The strangest aspect of this process is that it works. God *does* geometrize. Or, at least, humans often manage to convince themselves that She does. The growth of scientific understanding has been matched pace for pace by the development of mathematics; the two go hand in hand. The image of the Geometer God is powerful in art: for example, William Blake's painting *The Ancient of Days* of 1794 (Figure 1.1) shows the deity, dividers in hand, measuring up the universe for the act of creation.

The precursor to this book, *Does God Place Dice?*, examined *chaos*, the new mathematics of irregularity – the (sometimes only apparent) *absence* of pattern. *Fearful symmetry* is a 'prequel' rather than a sequel: it centres around the mathematics of regular pattern, which is conceptually simpler than chaos. Patterns in nature are a constant source of surprise and delight, and the dominant source of pattern in symmetry.

In everyday language, the words 'pattern' and 'symmetry' are used almost interchangeably, to indicate a *property* possessed by a regular arrangement of more-or-less identical units – for example the typographical pattern EEEEEEEEEEEE. Nature uses a similar pattern to design centipedes. Mathematicians use the word 'pattern' informally, and in much the same way; but they reserve 'symmetry' for a concept that is more precise than its everyday usage, and in some respects slightly different from it. The mathematician's view of symmetry applies to an idealized string of Es that is infinite both to left and right. It focusses upon the *transformations* that leave the entire string of symbols looking exactly the same. One such transformation is 'shift every symbol one place sideways'. Another is 'turn everything upside down'. The first transformation expresses the fact that each segment of a centipede looks the same as its neighbours; the second, that a centipede looks exactly the same in a mirror. There are other transformations that leave the entire string of symbols looking

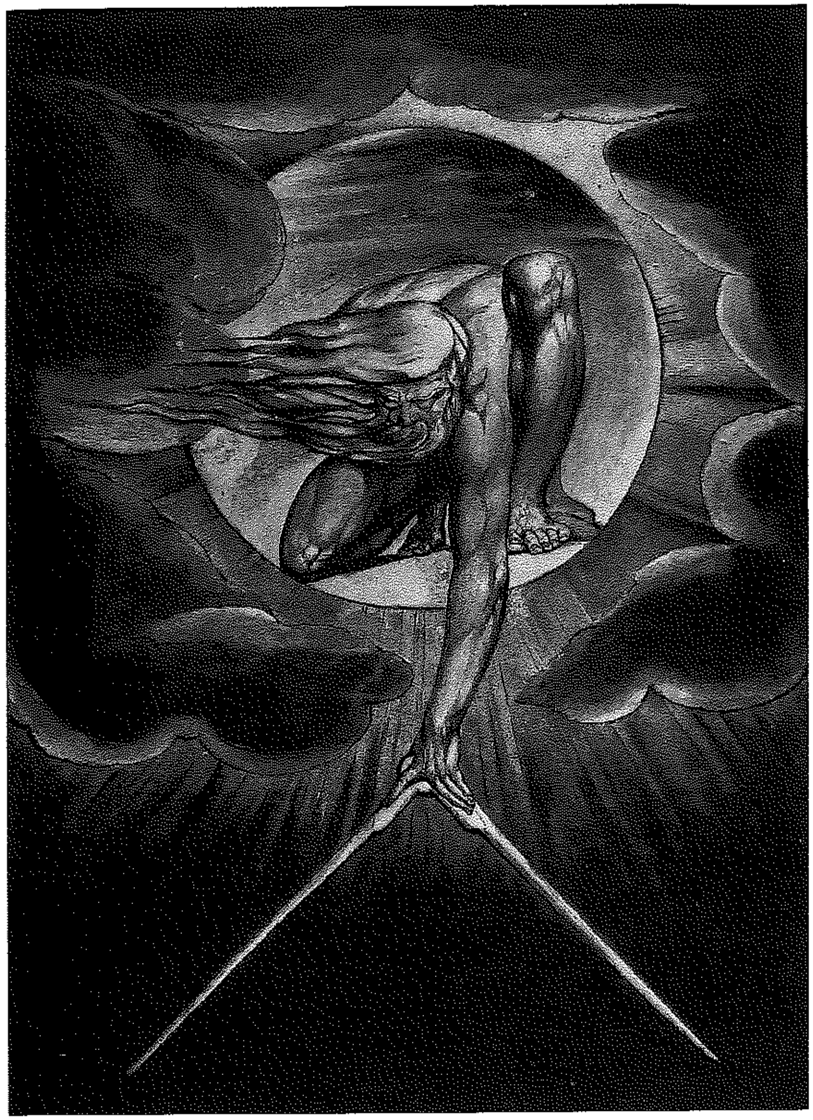

Figure 1.1 The Geometer God plans His universe: William Blake's The Ancient of Days

exactly the same, such as 'shift two places sideways and turn upside down', but they are simply combinations of the two just described.

To a mathematician, an object possesses symmetry if it retains its form after some transformation. A circle, for example, looks the same after any rotation; so a mathematician says that a circle is symmetric, even though a circle is not really a pattern in the conventional sense – something made up from separate, identical bits. Indeed the mathematician generalizes, saying that *any* object that retains its form when rotated – such as a cylinder, a cone, or a pot thrown on a potter's wheel – has *circular symmetry*. One advantage of this approach is that it provides a quantitative approach to symmetry, allowing comparisons between different objects. For instance, an object that retains its form under only *some* rotations – such as a square, which loses its orientation unless it is rotated through a multiple of a right angle – can sensibly be described as being *less* symmetric than a circle.

One of the great themes of the past century's mathematics is the existence of deep links between geometry and symmetry. Blake's painting anticipates this in its powerful suggestion of bilateral symmetry. If the signature of a Dicing Deity is chaos, then the signature of a Geometer God is symmetry. Is it inscribed upon our world?

Indelibly. Our universe is an apparently inexhaustible source of symmetric patterns, from the innermost structure of the atom to the swirl of stars within a galaxy. Plato made mathematical regularity the keystone of his philosophy, and viewed reality as an imperfect image of an ideal world of pure forms. In Book VII of *The Republic* he offers a striking allegory of people imprisoned in a cave, able to see only shadows of the outside world on the wall:

> And do you see, I said, men passing along the wall, some apparently talking and others silent, carrying vessels, and statues and figures of animals made of wood and stone and various materials, which appear over the wall?
>
> You have shown me a strange image, and they are strange prisoners.
>
> Like ourselves, I replied; and they see only their own shadows, or the shadows of one another, which the fire throws on the opposite wall of the cave?
>
> True, he said; how could they see anything but the shadows if they were never allowed to move their heads?
>
> And of the objects which are being carried in like manner they would only see the shadows?
>
> Yes, he said.
>
> And if they were able to talk with one another, would they not suppose that they were naming what was actually before them?

Whether or not we embrace Plato's theory of forms – ideal shapes whose 'shadows' we see on the walls of our world, imagining them to be the primary reality – there are few of us who are unmoved by the extraordinary tendency of nature to produce mathematical patterns. How do such patterns arise? We shall concentrate on one very fundamental process of pattern-formation, known as *symmetry-breaking*. This is itself a paradoxical phenomenon: it occurs when a symmetric system starts to behave less symmetrically. In some manner – which we'll explore – symmetry gets lost. Curiously, the typical result of a loss of symmetry is pattern, in the sense of regular geometric form, because only seldom is *all* symmetry lost. An oddity of the human mind is that it perceives *too much* symmetry as a bland uniformity rather than as a striking pattern; although some symmetry is lost, pattern seems to be gained because of this psychological trick. We are intrigued by the pattern manifested in circular ripples on a pond (why circles?), but not by the even greater symmetry of the surface of the pond itself (it's pretty much the same everywhere). Mathematically, a uniform, featureless plane has a vast amount of symmetry; but nobody ever looks at a wall painted in a single colour and enthuses over its wonderful patterns. This perceptual quirk often makes nature's patterns seem more puzzling than they really are.

To those whose perceptions have been suitably sharpened, symmetry-breaking is everywhere. It's visible in the way people walk and horses trot, in the dew that glistens on spiderwebs at dawn, the ripples on a pond, the swell of ocean waves, the stripes of a tiger, the spots of a leopard. It evidences itself in less homely ways: the glittering facets of crystals, the ponderous vibrations of stars, the spiral arms of galaxies – perhaps even the gigantic voids and clusters of the universe, of whose presence science has only recently become aware.

Or so we shall argue.

Even the splash of a raindrop is symmetry-breaking in action. Let's begin with raindrops.

The Shape of a Splash

Our favourite oddball science book is *On Growth and Form* by d'Arcy Thompson. If you've never read this provocative and penetrating treatise, get a copy from somewhere – though be warned, part of its appeal is an outmoded charm, so don't take it too seriously. Thompson was a pioneer of the idea that there are mathematical features to biological form. Prominently displayed at the very front of his book there's a wonderful and slightly disturbing picture of a drop of milk hitting the surface of a bowl, filled with the same liquid; the splash is frozen by high-speed photography, for us to contemplate at leisure. When raindrops hit a puddle, or inkblots hit paper, they must do something similar. Have you ever wondered what shape a splash is?

It looks like a crown.

From the point of impact rises a smooth, circular ring, surprisingly thin-walled, curving gracefully outwards as it rises. But the ring doesn't remain circular: it breaks up into 24 pointed spikes (Figure 1.2). Why does it break up? Why 24? These are good questions. The spikes are (almost) regularly spaced. Why? That's another good question. For a time we'll accumulate questions; eventually we'll attempt to answer some of them. The spikes come to a sharp point; most have just thrown off a tiny rounded droplet of milk (why?), and the rest are about to. You can see traces of such spikes in the way water is flung out when raindrops hit a puddle. And you know inkblots are always spiky, that's how cartoonists draw them. If somebody showed you a circular inkblot, you'd never recognize it; you'd think it was just a black circle. Which is curious, remarkably

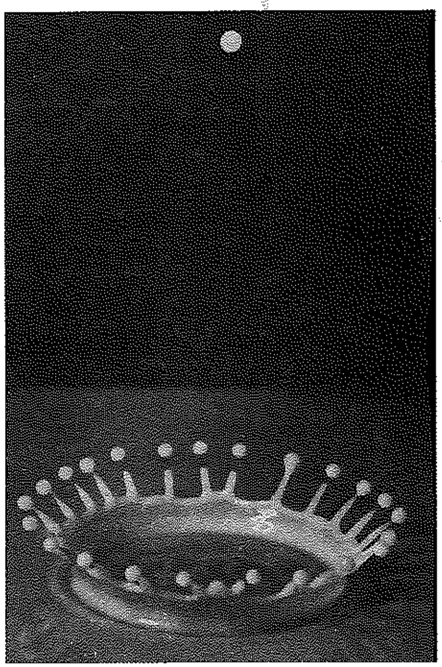

Figure 1.2 The symmetry of a splash

curious, because the drop of ink that produces the blot is (near enough) spherical, and the paper is flat. So what distinguishes the directions that lead to spikes from those that don't?

Focus your attention on the symmetry of the splash in d'Arcy Thompson's picture. It isn't perfect, but presumably that's due to slight imperfections in the shape of the original drop or the angle at which it fell. Maybe it was wobbling a little, maybe the milk in the bowl wasn't completely still. But the dominant feature, the spiky crown, doesn't look as though it's caused by such imperfections. You get the feeling that a *perfectly* spherical droplet would just give a *perfect* (and very probably also 24-pointed) crown! Let's assume this is true – for it is, in mathematical models of the process – and draw out the paradoxical consequences.

View the entire sequence of events from vertically above the bowl. A perfect droplet of milk has a perfect circular outline; and as it falls vertically downwards the outline remains circular. At all times, what you see has circular symmetry. If you now change viewpoint, and imagine your eye positioned at the centre of that circle, looking horizontally, then you won't be able to tell in which direction your eye is pointing. The droplet looks identical in any horizontal direction. So does the bowl, at least if we use a circular bowl and drop the milk into the middle of it. Thus we have a *cause*, the falling droplet; and the entire system of droplet, milk, and bowl has perfect circular symmetry – it looks the same in all horizontal directions.

What about the *effect* – the splash?

That *doesn't* have circular symmetry. It looks different, depending on which direction you view it from. Imagine placing yourself so that its nearest part is a spike; now walk round a few degrees, and the nearest part is a gap instead. The two views are similar, but not identical: where one has spikes, the other has gaps.

Where did the symmetry go?

Curie's Principle

The great physicist Pierre Curie is best remembered for his work, with his wife Marie, on radioactivity, leading to the discovery of the elements radium and polonium. But Curie is also remembered for his realization that many physical processes are governed by principles of symmetry. In 1894, in the *Journal de Physique Théorique et Appliquée*, Curie gave two logically equivalent statements of a general principle from the folklore of mathematical physics:

- If certain causes produce certain effects, then the symmetries of the causes reappear in the effects produced.
- If certain effects reveal a certain asymmetry, then this asymmetry will be reflected in the causes that give rise to them.

Curie's Principle (we'll refer to it in the singular, since its two statements are equivalent) is more subtle than it seems: like the utterances of politicians, its truth depends on how you interpret it. Let's begin with the simplest interpretation, which we can paraphrase as 'symmetric causes produce equally symmetric effects'.

At first sight, the principle is 'obviously' true. If a planet in the shape of a perfect sphere acquires an ocean, then we expect that ocean to be a perfect sphere as well, so it should coat the planet to the same depth everywhere. If the planet rotates, then we expect the ocean to bulge at the equator, but to retain circular symmetry about the axis of rotation. The flow of air past a symmetric obstacle will be symmetric. A rubber cube compressed by equal forces perpendicular to each face will obviously remain a cube, albeit a smaller one. The flow of fluid in an apparatus with circular symmetry will have circular symmetry. Isn't that right? After all, what else could happen?

Let's examine one example in more detail: the flow of air past an obstacle – in this case, a jumbo-jet. In the Kensington Science Museum in London there's an engineering model of a jet airliner, which during its design was used in a wind-tunnel to study the flow of air around the craft. An aircraft is bilaterally symmetric; that is, its left and right halves are mirror-images of each other. The engineers therefore only built half the model: they made a left-hand half of the plane and mounted it against a flat wall. The assumption behind this is that the flow of air past the aircraft must be bilaterally symmetric as well.

Here's a second example, also from industry. An American oil company was modelling the flow of oil in a system of production wells. It set up a mathematical model – a system of equations designed to mimic the main features of the real system – in which a hexagonal honeycomb of production wells extracted the oil, and injection wells at the centre of each hexagon pumped water in to displace it. As is common in industrial problems, a computer was used to set up and solve the mathematical model. In order to save computing time, the company assumed that the flow-patterns of the oil and water would have the same hexagonal symmetry as the wells; that let them restrict their analysis to a triangular region forming one twelfth of a hexagon (Figure 1.3).

Are the symmetry assumptions justified in these two cases?

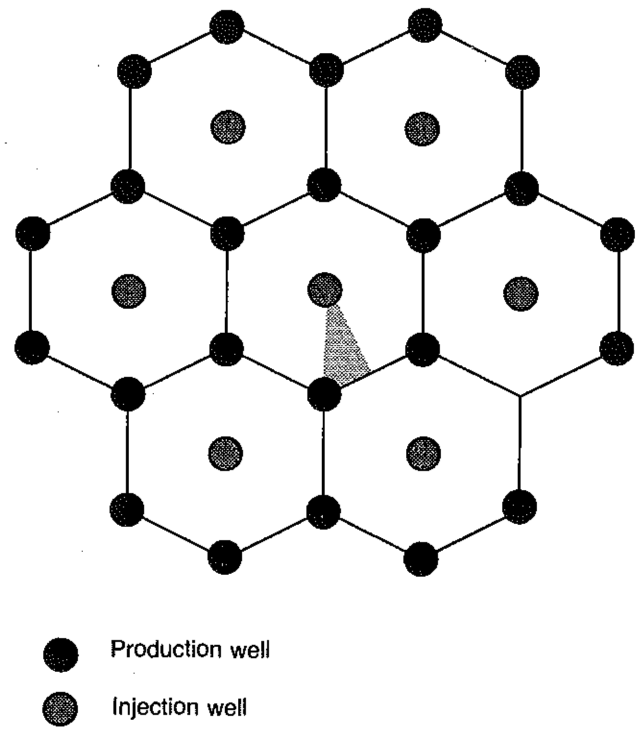

Figure 1.3 Hexagonally symmetric model of an oilfield. The flow of oil in the shaded triangle determines the rest of the flow 'by symmetry'. Or does it?

Let's think about the airliner. Imagine a complete airliner, both halves present, perfectly symmetric. Air flows past the craft according to the laws of aerodynamics. It's a mathematical fact, inherent in the form of those laws, that they are invariant under reflection. That is, if you reflect the aircraft, and simultaneously reflect the flow past it, then the result again obeys the laws. Now let's use the fact that the aircraft is symmetric under such a reflection: it means that after the reflection, the aircraft looks exactly the same as it did before. So the flow past it must look the same as before, too. But the flow past it has been reflected! We deduce that the flow must also be invariant under reflection.

A similar argument applies to the hexagonal array of oil wells. Hexagonal symmetry means that the array looks the same after various transformations: rotations through multiples of 60°, reflections in symmetry lines, translations sideways from one hexagon to the next. The laws of fluid flow imply that if the array of wells is transformed, and the flow-pattern is transformed in the same way, then it yields a physically valid flow. But the array of wells is symmetric under such transformations: it looks the same after applying them as it did before. So after each such transformation the flow past the array of wells must also look the same as it did before, implying that it's hexagonally symmetric too.

Convinced?

Curie Was Wrong

You shouldn't be. You've already seen an example for which Curie's principle – at least, in this interpretation – is false. The drop of milk. Remember the 24 spikes? Here the cause is circularly symmetric: drop, bowl, and milk all remain unchanged if you rotate everything through an arbitrary angle. But the effect, the splash, does *not* remain the same if you rotate it through an arbitrary angle. Some rotations, for example, move spikes to gaps, gaps to spikes. The only rotations that leave the crown unchanged are those that move spikes to spikes: rotations through multiples of $\frac{360}{24} = 15°$. The effect has less symmetry than the cause.

However, down at the bottom of the splash, when it first begins to rise from the milk, the effect *does* look circularly symmetric. so Curie's Principle seems to hold at first, but subsequently it goes wrong, as the initially circular symmetry breaks to 24-fold rotational symmetry. The splash captured by the high-speed camera isn't just an example of broken symmetry: it's a frozen representation of the entire *process* of symmetry-breaking. That's one reason why it's so intriguing.

Along with Curie, we're in trouble. We have a logical argument that seems to prove his principle is right, and an example that seems to prove it's wrong. Not such an unusual situation in scientific research! Either the example is wrong, or the argument is – maybe both, of course. Our job is to find out which. Here we can apply a direct approach: put the example up against the argument, and see which wins.

Here we go. The cause (droplet) has circular symmetry. That is, if we rotate everything through an arbitrary angle, nothing changes. Consider a splash (effect). Its form is derived from that of the droplet by applying the laws of fluid dynamics. Those laws are invariant under rotations; that is, if we rotate the drop through an arbitrary angle and rotate the splash through the same angle, we get a physically possible sequence of events. Seems watertight so far.

Now: circular symmetry implies that the rotated drop looks just like the original. Therefore the rotated splash looks just like the original.

But – *it doesn't*.

Hmm. There's a glitch in the logic, somewhere.

Let's think about it this way. Suppose we use computer graphics to make another photograph, but taken from a slightly different angle: to be precise, rotated through $7\frac{1}{2}°$, so that the spikes on the crown move to where the the gaps were in the original. If what we've been arguing so far is right, then this second picture represents a physical impossibility.

If you're happy with that, here's a question. How do you know that Figure 1.2 is the original picture from d'Arcy Thompson, as we've been assuming? Might it not be the fake, physically impossible computer picture? How can you tell? The only difference between the two is that what's North in one is $7\frac{1}{2}°$ East of North in the other.

But nobody's *marked* North ... and in a circularly symmetric system there's no reason to choose any particular direction to *be* North.

This is puzzling. A slightly rotated splash, with its spikes where once there were gaps, seems to be just as valid an effect as the splash that actually occurs. Can the droplet have *more than one effect*?

In the real world, no: something definite has to happen. You don't get two splashes at the same time. But in the mathematics, yes. Both splashes, the original and its rotation, are valid solutions to the same equations; valid consequences of the same physical laws. Instead of a single effect, we have a whole *set* of possible effects: all the different rotations of the 24-pointed crown. Our argument from symmetry doesn't prove that 'the' effect is symmetric under rotation: what it proves is that if *an* effect is rotated, then the result is a physically possible effect – but it could be a *different* one. The logical fallacy is the assumption that each cause produces a unique effect; the argument breaks down if several distinct effects are equally possible.

It may sound unlikely that several possible effects can arise from a single cause, but symmetric systems are like that. Symmetry means that, given some possible effect, all symmetrically related effects are also physically possible. Take the aircraft, for example. What the argument really shows is that, given any possible flow of air past the craft, its mirror image is also a possible flow (Figure 1.4). *They don't have to be the same.*

One of our acquaintances, an aircraft engineer, tells us that if you sit right at the back of one popular make of commercial airliner, then you'll notice that every five minutes or so it seems to twitch sideways a couple of feet. Five minutes later it twitches the other way. This is

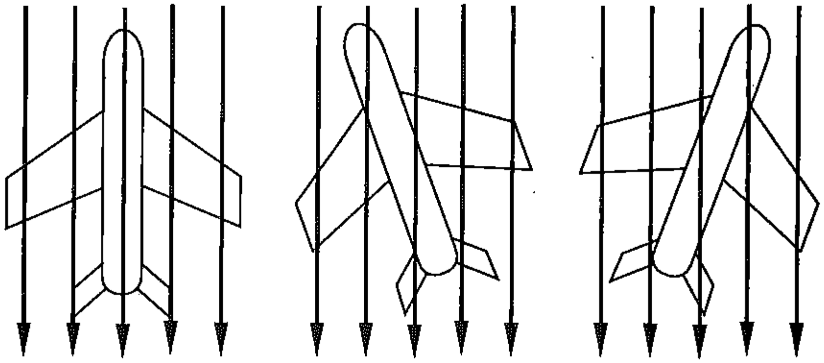

Figure 1.4 Flow past a bilaterally symmetric aircraft. (a) need not happen in practice; but if (b) does, so must (c)

because the flow of air past the aircraft is *not* bilaterally symmetric. The plane flies very slightly crabwise, its tail a foot or so to one side of the line of flight. Any change – a bit of turbulence, the automatic pilot adjusting the control surfaces – and it may twitch across to the symmetrically related position. The engineers who designed the aircraft know all about this effect. It's perfectly safe, and so tiny that it has no practical implications. You have to know what to look for before you can spot it happening. As a practical matter, *both* flows are extremely similar to each other, and to the perfect symmetric flow that the wind-tunnel experiment, by using half an aircraft, forces to occur. But it's an interesting case where our 'Curie Principle' intuition about symmetry is violated, not only in theory, but in practice.

A final question which may be bothering you. It all seems a complicated way to design an aircraft. Why doesn't it just fly straight ahead?

It can't.

Loss of Stability

To see why an airliner may not be able to fly straight, we must explain what causes symmetry to break.

A simple, more familiar example is helpful at this point. Think of a perfect circular cylinder, say a tubular metal strut, being compressed by a force. What happens? Nothing much at first, but if the force becomes sufficiently large, then the strut will buckle. The buckling is

not a consequence of lack of symmetry caused by the force: even if the force is directed along the axis of the tube, preserving the rotational symmetry about that axis, the tube will still buckle. Buckled cylinders cease to be cylindrical – that's what 'buckle' means. Figure 1.5 shows the result of an experiment in which a metal cylinder, sandwiched inside a slightly larger glass one to prevent it buckling too far, is compressed from its ends by a uniform force. An elegant, symmetric pattern of identical dents appears, in a fairly random order.

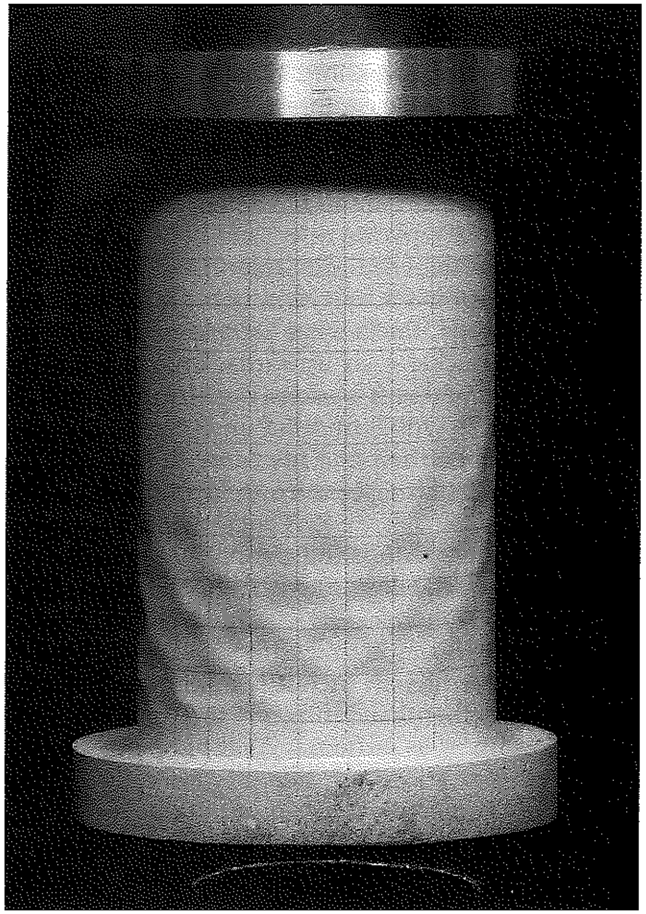

Figure 1.5 Buckling of a cylinder into a symmetric pattern of dimples, under pressure from the ends. The cylinder is set inside a slightly larger glass one to prevent large buckles

Mathematically, even when it is compressed by very large forces, a cylindrical strut can remain cylindrical. There does exist a solution to the mathematical model in which there is no buckling. But mathematically, you can balance a chain on end, provided each link sits precisely above the one below it. The Indian rope trick notwithstanding, this is impossible in the real world. So mathematical *existence* of solutions to equations isn't enough.

The missing ingredient is *stability*. Natural systems must be stable; that is, they must retain their form even if they're disturbed. A pin lying on its side is stable and can occur in the real world. A pin balanced on end is *theoretically* possible, but it does not occur in the real world, because it is unstable: the slightest breath of wind and it topples. The same is true of a vertical chain, and of an unbuckled strut under enormous compression. What actually happens when a strut buckles is that when the compression reaches a critical load, the unbuckled state becomes unstable. The strut then seeks a nearby stable position; and that's buckling.

Here's an experiment for you to try, either with real apparatus or just in your mind. Take a plastic ruler, and hold it flat between your hands: then compress the ends. The ruler has top/bottom symmetry (provided it's flat), and in its fully symmetric state it's horizontal. If you don't press too hard, that's what happens. But greater compression makes the ruler buckle – say upwards. By symmetry there ought also to be a position where it buckles downwards; and there is. If you've got three hands, you'll find that in between the two buckled states there's a position where the ruler *tries* to be horizontal, in the sense that it takes very little pressure to keep it there; but it keeps flipping away when you release it. That's the fully symmetric state again, but now it's unstable. As the compressive forces increase, the symmetric state loses stability, and the ruler has to buckle to an asymmetric position.

The same is true of the crabwise-flying aircraft: straight-ahead flight is unstable. If it tilts an inch away from straight, it tends to move further away, rather than returning, thanks to the very complex interplay of forces upon its airframe. As it happens, there's a stable position about a foot to either side of the centre, thoughtfully provided by the design engineers.

We can also see why the milk splash starts out having circular symmetry, but then loses it, forming spikes. We've already observed that initially the growing ring of milk has a circular form, and that the spikes appear higher up. Presumably the ring becomes unstable when it grows too high, and it – buckles. Just like the sphere and cylinder, it goes wavy. The detailed fluid dynamics must select the form with 24 waves, though we can't confirm that without doing a very nasty calculation. However, by using the general mathematics of symmetry-breaking, we *can* predict that the remaining symmetry will be that of a regular polygon, although the number of sides – here 24 – can't be deduced from symmetry-breaking alone. In short, while the detailed structure of the crown is a surprise, its general features, especially its symmetries, are not.

Curie was right in asserting that symmetric systems have symmetric states – but he failed to address their stability. If a symmetric state becomes unstable, the system will do something else – and that something else need not be equally symmetric.

Curie Was Right

We shouldn't be too pleased with ourselves, having caught a Nobel-winning physicist making a mistake: we may just have misunderstood what he was trying to tell us. There's a disturbing gap in the story at this point. We've said that *mathematically* the laws that apply to symmetric systems can sometimes predict not just a single effect, but a whole set of symmetrically related effects. However, Mother Nature has to *choose* which of those effects she wants to implement. How does she choose?

The answer seems to be: imperfections. Nature is never *perfectly* symmetric. Nature's circles always have tiny dents and bumps. There are always tiny fluctuations, such as the thermal vibration of molecules. These tiny imperfections load Nature's dice in favour of one or other of the set of possible effects that the mathematics of perfect symmetry considers to be equally possible.

Take a perfect sphere, and compress it with a uniform radial force. That's a fancy way to say 'try to squash a ping-pong ball'. When the sphere buckles, it develops a dent somewhere. According to the mathematics, the dent is circular in shape, and indeed the buckled sphere retains circular symmetry about some axis (Figure 1.6). Don't expect to see anything as perfect as this if you squash a real ping-pong ball: the picture is an ideal case. In principle, if a buckling pattern can occur, than any rotation of that pattern can also occur. That is, the axis of rotational symmetry, and the dent at one end of it, can be in any place you like. The *shape* of the buckled sphere is the same in each case, but its *position* is not. Not only is Figure 1.6 a possible form for a buckled sphere: so is anything you can get by rotating it. A real sphere, however, is never exactly spherical. It may, for example, be slightly thinner at one place than at another. The dent

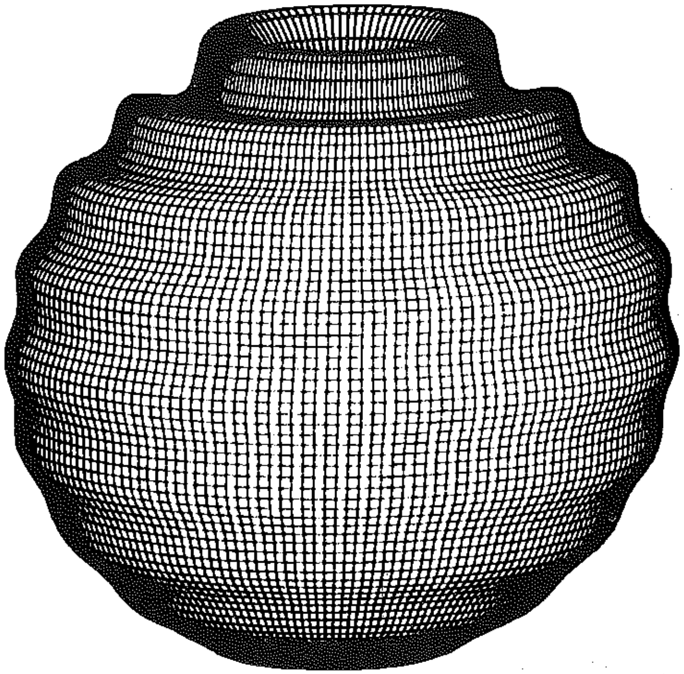

Figure 1.6 When a sphere first buckles, it has circular symmetry

is more likely to occur at such a weak spot. That's how Mother Nature loads the dice.

In other words, the issue is the relation between a mathematical model and the reality that it is supposed to represent. Nature behaves in ways that look mathematical, but nature is not the same as mathematics. Every mathematical model makes simplifying assumptions; its conclusions are only as valid as those assumptions. The assumption of perfect symmetry is excellent as a technique for deducing the conditions under which symmetry-breaking is going to occur, the general form of the result, and the range of possible behaviour. To deduce exactly *which* effect is selected from this range in a practical situation, we have to know which imperfections are present.

In this sense, Curie was absolutely right. If we see a ping-pong ball with a dent in one side, we are right to deduce that something happened to it that was not spherically symmetric; that the asymmetric effect we observe must have had an asymmetric cause. However,

that asymmetry might be just a tiny fluctuation in an otherwise perfectly symmetric setting. So whether Curie's Principle is true depends on what questions we ask.

History is littered with examples where scientists and philosophers have misapplied Curie's Principle, seeking *large-scale* asymmetries in causes, to account for large-scale asymmetries – such as patterns – in effects. For example, until very recently astronomers thought that the spiral arms of galaxies were caused by magnetic fields. As we'll see in chapter 6, they're beginning to think that the spirals are the result of gravitational symmetry-breaking. But Curie's Principle doesn't say that the size of the asymmetry is comparable in the cause and the effect. As we shall see, when systems have symmetry, there's a good chance that the symmetry may break. When it does, very tiny asymmetries play a crucial role in selecting the actual outcome from a range of potential outcomes.

Trucks and Trees

Once you've become sensitized to the possibility of symmetry-breaking, you see it everywhere. A few summers ago one of us (INS) was driving along a freeway in up-state New York. Ahead was a large truck, with two mud-flaps at the rear. They were flapping - as all good mud-flaps should. But they weren't flapping in unison. When the left-hand flap was moving forwards, the right-hand one was moving backwards, and *vice versa*. An engineer would note that the oscillations were 180° out of phase. A physicist would observe that the oscillations were caused by vortex-shedding: the truck was leaving a train of tiny tornados in its wake, peeling off in turn to the left and the right, and wiggling the flaps as they passed. But what your humble author saw was an example of symmetry-breaking. The truck and its arrangement of flaps was, near enough, left-right symmetric; but the motion was asymmetric: the left-hand flap and the right-hand flap were not performing identical motions.

In fact the pattern of vortices has its own symmetry, but of a different kind from that of the truck that produces it. The truck is symmetric under a reflection that interchanges left and right (Figure 1.7a); the vortex train that it sheds is symmetric under a *glide reflection*, which interchanges left and right but also moves a suitable distance in the direction of the truck's motion (Figure 1.7b).

Here's another broken symmetry, witnessed by the same author on a previous trip. In northern California grow huge trees, redwoods and sequoias. The trunk of a tree is approximately cylindrical, and we

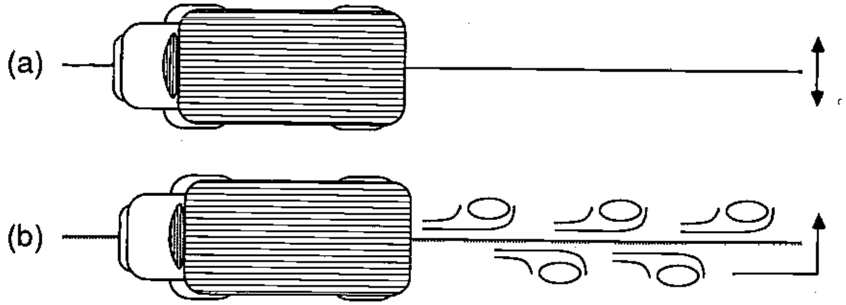

Figure 1.7 (a) Bilateral symmetry of a truck. (b) Glide-reflection symmetry of the vortex train that it produces

may assume that it has exact cylindrical symmetry. The symmetries of a cylinder are of three kinds: rotation, translations, and reflections. If you rotate a cylinder about its axis (Figure 1.8a) it looks exactly the same as before; and the same is true if you translate it in the direction of its axis (Figure 1.8b). To be precise, translational symmetry holds only for an infinitely long cylinder; but it is valid to a good approximation for a sufficiently long one. There are also two distinct types of reflectional symmetry (Figures 1.8c and 1.8d): either in a vertical mirror or in a horizontal one.

It seems plausible that the pattern of bark on the tree should have similar symmetry to the tree itself. Now a pattern of bark with (a good

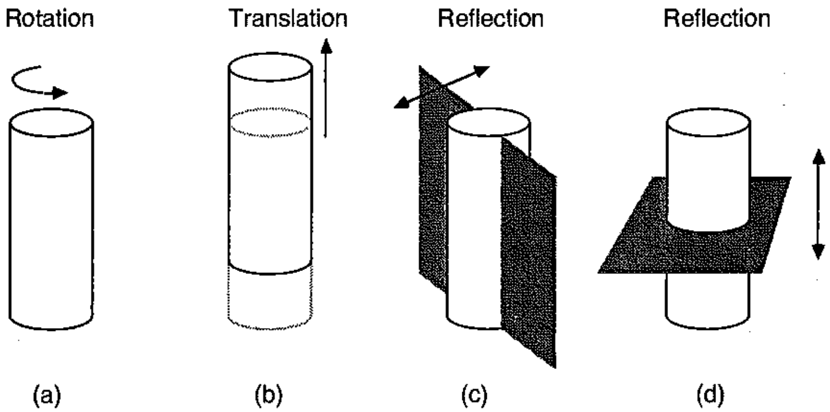

Figure 1.8 Symmetries of a cylinder

approximation to) full cylindrical symmetry will have to look pretty much the same after any of those rotations, translations, and reflections. That means it should resemble Figure 1.9a, with the grooves in the bark running roughly vertically – as they do on most trees.

But on *some* of the Californian trees, you'll see a *spiral* pattern in the bark (Figure 1.9b). The spiral still has some symmetry, but of a different kind. If you rotate a helical spiral, *and* translate it, then it looks the same. So the symmetry of a spiral is a mixture of rotation and translation, known as a *screw*. Indeed, this is the reason why a carpenter's woodscrew works: as it rotates *and* goes deeper into the wood (or translates) it fits into the same hole. Real screws are tapered so they enlarge the hole slightly as they go, for a tight fit; but a bolt

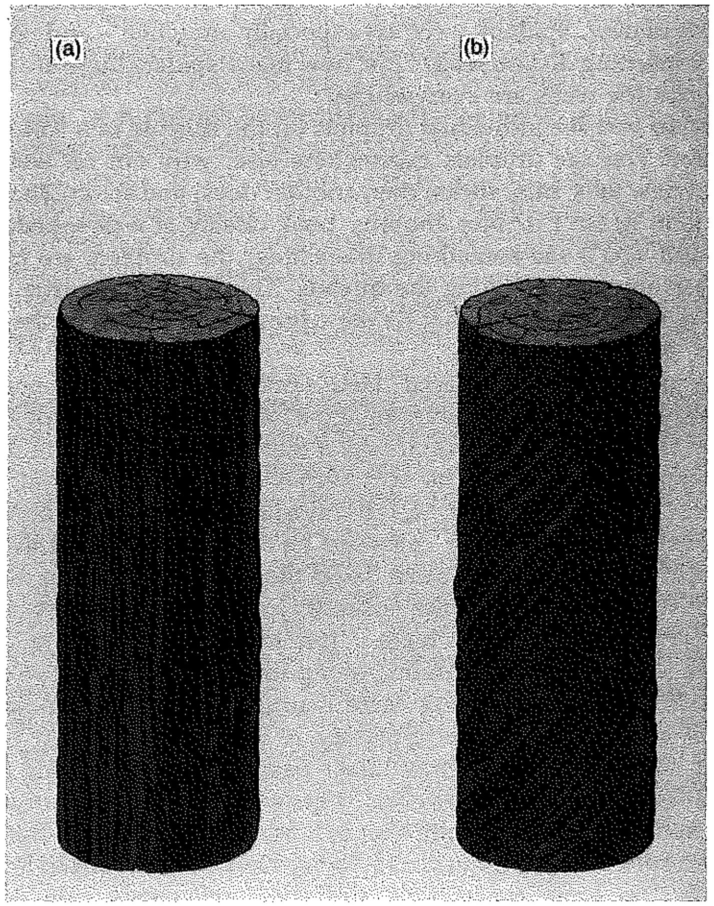

Figure 1.9 (a) Straight bark on a sequoia. (b) Spiral bark

with a helical thread has exact symmetry of this kind. Did something strange happen to those trees that developed spiral bark? Pesticides, a bad winter, drought? Or should we *expect* spiral patterns as well as perfectly symmetric ones? From our new-found viewpoint, there's nothing surprising about trees with spiral patterns to their bark. If the perfectly symmetric pattern represents an unstable development, then tiny disturbances will cause the symmetry to break. Spirals are one of the common ways to break cylindrical symmetry, so a spiral pattern could very well develop instead.

Spiderwebs

If you get up early on a spring morning, and go outside, everything sparkles like a jeweller's shop. The sunlight reflects from countless droplets of dew. Spiders' webs are especially attractive – and only on a dewy morning do you realize just how many spiders there are in the world, for the webs are everywhere.

Why do you see individual droplets on the threads of a web? Silly question! You see droplets everywhere else, why not on webs? A drop of dew falls, hits, sticks; then another does, and so on. That's all there is to it, isn't it? Not entirely. Look closely at a dew-bedecked web, and you'll see something interesting: the droplets are (pretty much) evenly spaced along the threads. That's especially true along the long straight threads that tie the edges of the web to bushes and walls.

Why are they equally spaced? Here's one explanation, the one that appeals to symmetry-breaking. Imagine an idealized thread of spider silk, an infinite, straight line. Imagine that dew falls on to the thread and spreads, coating it. The symmetries of the thread are all rigid motions of the line; that is, all translations and all reflections. Moreover, it has a circular cross-section. If symmetry isn't broken, then the coating of dew must have the same symmetries as the thread; which means that the entire thread will be coated evenly in water, and the dew will create a wet cylinder, of uniform cross-section, with the thread running along its centre. Why doesn't that happen? The fully symmetric state is unstable, because the surface tension of the water is compressing it along the direction of the thread, like a very tall stack of books tied with elastic. So the water bulges here and there. However, it generally seems to be true that when symmetry breaks, quite a bit of it tends to be retained. Both the buckled sphere and the buckled cylinder have a lot of pattern left, for example. So some translations and some reflections may not be broken. If a translational symmetry remains, then the entire pattern

of dewdrops must repeat at regular intervals; if there's a reflectional symmetry, then those component patterns will be bilaterally symmetric. And that's what we find: regularly spaced blobs, nearly spherical and each blob left-right symmetric.

We could spend some considerable time describing the physics of this situation, and fleshing out the abstract symmetry-breaking 'explanation'. However, we just want you to look at dew-bedecked spiderwebs with fresh eyes, to see them as yet another example of pattern formation caused by broken symmetry. So we'll move on.

Honeycomb Lakes

In some parts of the world there are curious flat mounds of rocks – geologists call them stone nests – arranged in a roughly hexagonal pattern like a honeycomb (Figure 1.10). Why? Originally the stones were on the bed of a large, shallow lake. The Sun's rays heated the lake, giving rise to currents in the water. Now a large lake is approximately symmetric under translations in any direction, as well as rotations. If no symmetry were broken, the flow of water would also be symmetric under all translations and rotations – which means no flow at all! All that would happen is that the heat would be *conducted* through a stationary lake.

To find out what really happens, take a frying-pan filled with a shallow layer of water (the lake) and sit it on the stove (Sun). The

Figure 1.10 Hexagonal pattern of stone nests, caused by convection

depth of water should be no more than half a centimetre or so. Turn on the stove and heat the pan *very* gently. (Be careful, and if you are under the age of 21 or cohabiting with another human being ask permission first. A cast iron pan is best, to distribute the heat uniformly across the base of the pan. It may be wiser to treat this as a thought experiment: at any rate, don't expect too spectacular a result – proper laboratory equipment works better.) What happens?

You get strange cellular patterns.

The physical reason is that hot fluid tends to rise. As the temperature increases, the hot layer on the bottom is trapped beneath a layer of colder, denser fluid. This situation is unstable, and is destroyed by the onset of convection. Hot fluid rises in some regions, cold fluid descends in others. A cellular pattern of moving fluid, known as Bénard cells after the French scientist who discovered them, appears. Sometimes the pattern consists of parallel rolls, sometimes it is a honeycomb array of hexagons. In a real pan the symmetry is approximate and the pattern is rather irregular, but in an idealized infinite pan you get perfect honeycombs (Figure 1.11). These patterns still have a great deal of symmetry, but less than that of the pan.

Stone nests are relics of a similar process occurring in the lake. Over long periods of time, convection cells in the water clump the stones together: the approximate hexagonal pattern of the nests reflects that of the convection cells.

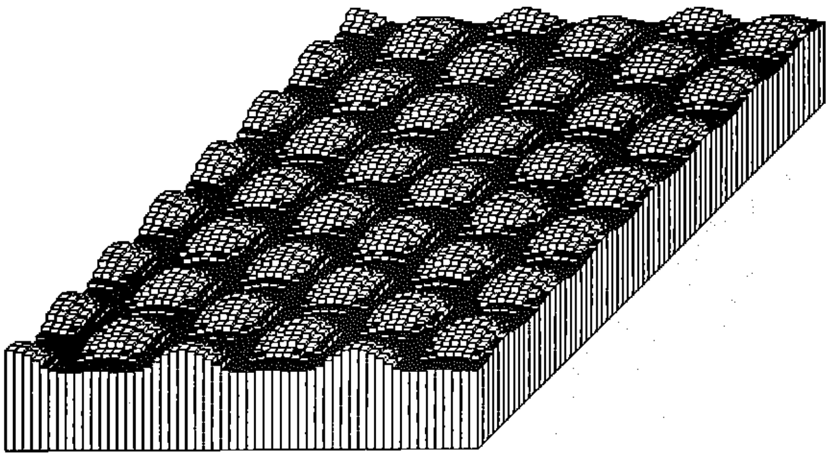

Figure 1.11 Computer-generated honeycomb pattern in a mathematical model of convection. The height of the surface represents the vertical speed with which fluid is moving

The truth is thus more complex than Curie's Principle suggests. A symmetric system will take up an equally symmetric state, *except when it doesn't!* It might seem that this just says that Curie was right except when he was wrong, but by analysing the conditions under which symmetry-breaking occurs we can give the idea some genuine content. To do that, we need to make the idea of symmetry precise, and that we do in the next chapter.

Corn Circles

There's a moral to the tale, and here's as good a place as any to make it. Our discussion of Californian tree bark doesn't *explain* how the spiral develops: we still don't know if it's caused by climate, genetics, a virus, or whatever. But we've understood an important point: spirals are a likely development, whatever the cause. If you want to find the cause, don't get too obsessed with spirals!

California is a place of extreme beauty, but also a place of weird cults and lifestyles. It's not the only place with that problem. As we write, the British media are awash with bizarre tales of 'corn circles'. These are remarkable depressions that occur, apparently spontaneously, in fields of standing corn (Figure 1.12). The simplest version is perfectly circular; other more elaborate shapes include concentric rings, circles with four smaller circles arranged around them, equally spaced, or entire chains of circles. The circles generally appear on still nights in large, flat fields.

Figure 1.12 Hoax, UFO, or broken symmetry? The enigma of corn circles

There are lots of theories. One is that they're the imprints of UFOs – after all, UFOs are circular, aren't they? Nobody has yet explained what benefit aliens get from landing in a cornfield, though. Another is that they're the result of vortices in still air – like smoke-rings. Smoke-rings are naturally circular. A third is that they're the result of electrostatic charges building up in the heads of corn, causing it to collapse like a tall stack of dominoes when you press too hard on the top. A fourth is that they're hoaxes, the lads from the local Young Farmers' Club out on a drunken spree: all you need is a peg and a rope, and you can soon trample a circle.

The interesting feature of almost all discussion of these phenomena is that they always focus on the circular shape. How could natural causes produce something as regular as a circle? It's always seen as the key to the enigma.

If you know about symmetry-breaking, however, you soon realize that the circularity is the one thing that *doesn't* need explanation! Think of it this way. A large, flat field is a very symmetric thing: it's not far removed from that mathematical ideal, an infinite plane. If the boundaries of the field are far away, the local geometry doesn't alter if you translate the field sideways a bit, reflect it in some line, or *rotate* (aha!) it. This is true even more of the corn: modern agricultural techniques lead to considerable uniformity in height, strength, and yield.

Suppose something happens to cause this symmetry to break. If the source of asymmetry is something that occurs at a point – for example, it may be nothing more than a bare patch of rocky ground – then the translational symmetry will be broken; but the rotational symmetry will remain. There need not even *be* a source at all: if the uniform state becomes unstable, symmetry-breaking will occur, and rotational symmetry is a strong possibility. What's the simplest geometric form with rotational symmetry?

A circle.

In symmetry terms, the process is completely analogous to the creation of circular ripples on a pond, by throwing in a stone. The pond has the symmetry of an infinite plane; the stone breaks symmetry at a point. The physical mechanisms are probably quite different, of course; and (so far) nobody has made a fortune writing books claiming that ripples on ponds are evidence of alien visitations.

The explanation of corn circles via electrostatic forces takes off from this point. The explanation by vortices applies similar reasoning, not to the field of corn, but to the atmosphere above it, although its devotees don't express it in terms of broken symmetry. The sharp edges of the circles are also no mystery: it takes a very specific

amount of force to break a stalk of corn, and with modern farming techniques, stalks of corn are almost identical, so it takes the identical force to break them all. The boundary between regions where the force (be it of atmospheric, electrostatic, or extraterrestrial origin) is large enough to break the corn, or not, is of necessity sharp. Indeed when storms lay waste to cornfields, they tend to flatten it in well-defined, though irregular, areas. Having said all this, without doubt some circles are due to the efforts of the Young Farmers' Club, especially now that the phenomenon is attracting such coverage. An explanation of why aliens find it so amusing to land in cornfields remains elusive – maybe ET is trying to phone home with a very large circular satellite dish.

What have we learned by applying the ideas of symmetry-breaking? That the circular form is inherent in the flat, uniform nature of the surrounding field. That the same psychological trick is twisting our perceptions: we see the symmetry of the circle, we ignore the even greater – but blander – symmetry of the field of corn, and wonder where the circles come from. Corn circles are probably no more enigmatic than pond circles. The difference is that we can't see the stone being flung in, and we haven't yet worked out the details of the physics. Even so, we've learned something very valuable indeed: *not to be unduly impressed by the fact that corn circles are circular.*

If only the news media would do the same.