Volume One of

THE WORLD OF MATHEMATICS

A small library of the literature of mathematics from A'h-mosé the Scribe to Albert Einstein, presented with commentaries and notes by JAMES R. NEWMAN

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in sculpture, painting, architecture, ornament and design; its manifestations in organic and inorganic nature; its philosophical and mathematical significance. Symmetry establishes a ridiculous and wonderful cousinship between objects, phenomena and theories outwardly unrelated: terrestrial magnetism, women's veils, polarized light, natural selection, the theory of groups, invariants and transformations, the work habits of bees in the hive, the structure of space, vase designs, quantum physics, scarabs, flower petals, X-ray interference patterns, cell division in sea urchins, equilibrium positions of crystals, Romanesque cathedrals, snowflakes, music, the theory of relativity. The structure of these relationships is depicted by Weyl in a remarkable sweep. The style is not always easy; neither is the subject. Nevertheless the book affords an entry into a profound and fascinating subject which demonstrates, perhaps uniquely, the working of the mathematical intellect, the evolution of intuitive concepts into grand systems of abstract ideas. I have selected the first two of the lectures—on bilateral and related symmetries; I was tempted to give the entire series. You will discover within a few pages why it was so hard to resist the inclination.

. . . What immortal hand or eye,
Dare frame thy fearful symmetry?

—William Blake

9 Symmetry

By HERMANN WEYL

BILATERAL SYMMETRY

IF I am not mistaken the word *symmetry* is used in our everyday language in two meanings. In the one sense symmetric means something like well-proportioned, well-balanced, and symmetry denotes that sort of concordance of several parts by which they integrate into a whole. *Beauty* is bound up with symmetry. Thus Polykleitos, who wrote a book on proportion and whom the ancients praised for the harmonious perfection of his sculptures, uses the word, and Dürer follows him in setting down a canon of proportions for the human figure.[1] In this sense the idea is by no means restricted to spatial objects; the synonym "harmony" points more toward its acoustical and musical than its geometric applications. *Ebenmass* is a good German equivalent for the Greek symmetry; for like this it carries also the connotation of "middle measure," the mean toward which the virtuous should strive in their actions according to Aristotle's Nicomachean Ethics, and which Galen in *De temperamentis* describes as that state of mind which is equally removed from both extremes: σύμμετρον ὅπερ ἑκατέρου τῶν ἄκρων ἀπέχει.

The image of the balance provides a natural link to the second sense in which the word symmetry is used in modern times: *bilateral symmetry,* the symmetry of left and right, which is so conspicuous in the structure of the higher animals, especially the human body. Now this bilateral symmetry is a strictly geometric and, in contrast to the vague notion of symmetry discussed before, an absolutely precise concept. A body, a spatial configuration, is symmetric with respect to a given plane E if it is

[1] Dürer, *Vier Bücher von menschlicher Proportion*, 1528. To be exact, Dürer himself does not use the word symmetry, but the "authorized" Latin translation by his friend Joachim Camerarius (1532) bears the title *De symmetria partium*. To Polykleitos the statement is ascribed (περὶ βελοποιϊκῶν, IV, 2) that "the employment of a great many numbers would almost engender correctness in sculpture." See also Herbert Senk, Au sujet de l'expression συμμετρία dans Diodore I, 98, 5–9, in *Chronique d'Egypte 26* (1951), pp. 63–66. Vitruvius defines: "Symmetry results from proportion . . . Proportion is the commensuration of the various constituent parts with the whole." For a more elaborate modern attempt in the same direction see George David Birkhoff, *Aesthetic measure*, Cambridge, Mass., Harvard University Press, 1933, and the lectures by the same author on "A mathematical theory of aesthetics and its applications to poetry and music," *Rice Institute Pamphlet,* Vol. 19 (July, 1932), pp. 189–342.

carried into itself by reflection in E. Take any line l perpendicular to E and any point p on l: there exists one and only one point p' on l which has the same distance from E but lies on the other side. The point p' coincides with p only if p is on E. Reflection in E is that mapping of space

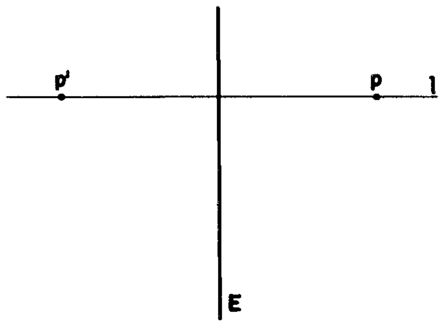

FIGURE 1—Reflection in E.

upon itself, S: $p \rightarrow p'$, that carries the arbitrary point p into this its mirror image p' with respect to E. A mapping is defined whenever a rule is established by which every point p is associated with an image p'. Another example: a rotation around a perpendicular axis, say by 30°, carries each point p of space into a point p' and thus defines a mapping. A figure has rotational symmetry around an axis l if it is carried into itself by all rotations around l. Bilateral symmetry appears thus as the first case of a geometric concept of symmetry that refers to such operations as reflections or rotations. Because of their complete rotational symmetry, the circle in the plane, the sphere in space were considered by the Pythagoreans the most perfect geometric figures, and Aristotle ascribed spherical shape to the celestial bodies because any other would detract from their heavenly perfection. It is in this tradition that a modern poet [2] addresses the Divine Being as "Thou great symmetry":

God, Thou great symmetry,
Who put a biting lust in me
From whence my sorrows spring,
For all the frittered days
That I have spent in shapeless ways
Give me one perfect thing.

Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.

The course these lectures will take is as follows.[3] First I will discuss

[2] Anna Wickham, "Envoi," from *The Contemplative Quarry*, Harcourt, Brace and Co., 1921.

[3] [The first two lectures are given here. Lecture 3 deals with ornamental symmetry, Lecture 4 with crystals and the general mathematical idea of symmetry. ED.]

bilateral symmetry in some detail and its role in art as well as organic and inorganic nature. Then we shall generalize this concept gradually, in the direction indicated by our example of rotational symmetry, first staying within the confines of geometry, but then going beyond these limits through the process of mathematical abstraction along a road that will finally lead us to a mathematical idea of great generality, the Platonic idea as it were behind all the special appearances and applications of symmetry. To a certain degree this scheme is typical for all theoretic knowledge: We begin with some general but vague principle (symmetry in the first sense), then find an important case where we can give that notion a concrete precise meaning (bilateral symmetry), and from that case we gradually rise again to generality, guided more by mathematical construction and abstraction than by the mirages of philosophy; and if we are lucky we end up with an idea no less universal than the one from which we started. Gone may be much of its emotional appeal, but it has the same or even greater unifying power in the realm of thought and is exact instead of vague.

I open the discussion on bilateral symmetry by using this

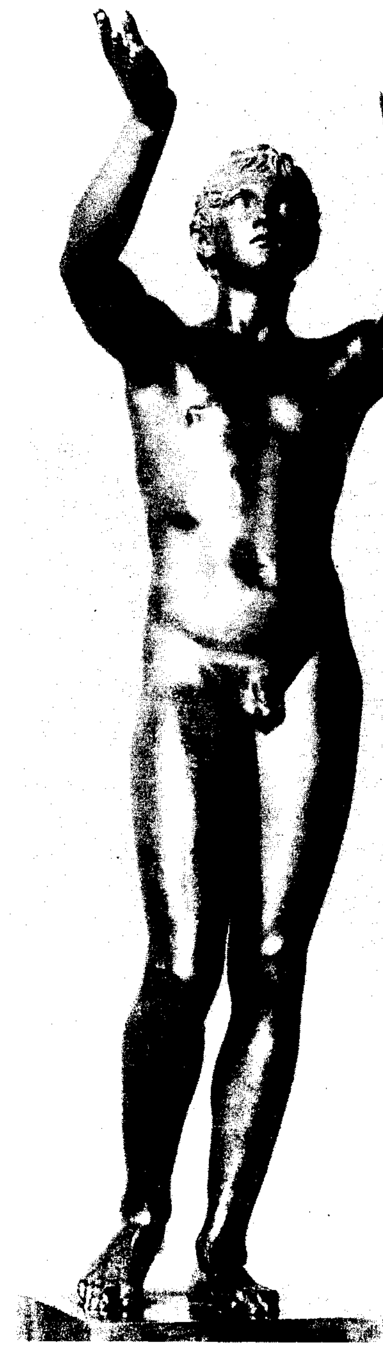

FIGURE 2

noble Greek sculpture from the fourth century B.C., the statue of a praying boy (Figure 2), to let you feel as in a symbol the great significance of this type of symmetry both for life and art. One may ask whether the aesthetic value of symmetry depends on its vital value: Did the artist discover the symmetry with which nature according to some inherent law has endowed its creatures, and then copied and perfected what nature presented but in imperfect realizations; or has the aesthetic value of symmetry an independent source? I am inclined to think with Plato that the mathematical idea is the common origin of both: the mathematical laws governing nature are the origin of symmetry in nature, the intuitive realization of the idea in the creative artist's mind its origin in art; although I am ready to admit that in the arts the fact of the bilateral symmetry of the human body in its outward appearance has acted as an additional stimulus.

Of all ancient peoples the Sumerians seem to have been particularly fond of strict bilateral or heraldic symmetry. A typical design on the famous silver vase of King Entemena, who ruled in the city of Lagash around 2700 B.C., shows a lion-headed eagle with spread wings *en face*, each of whose claws grips a stag in side view, which in its turn is frontally

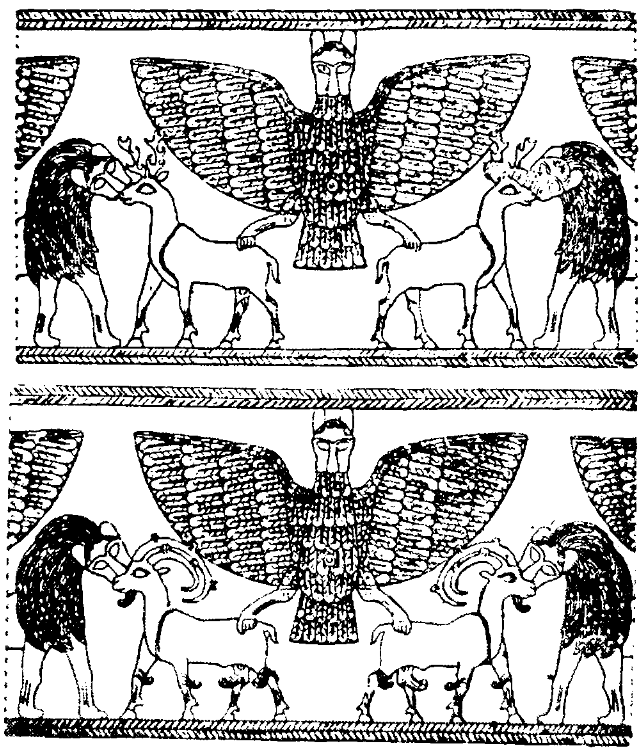

FIGURE 3

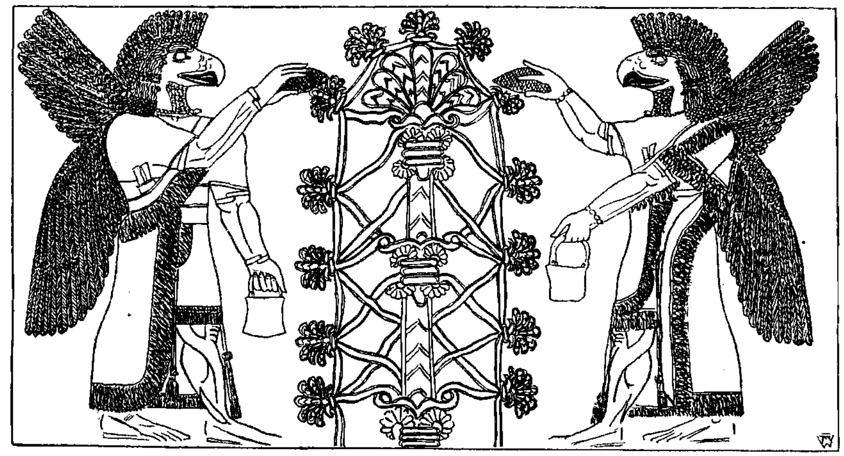

FIGURE 4

attacked by a lion (the stags in the upper design are replaced by goats in the lower) (Figure 3). Extension of the exact symmetry of the eagle to the other beasts obviously enforces their duplication. Not much later the eagle is given two heads facing in either direction, the formal principle of symmetry thus completely overwhelming the imitative principle of truth to nature. This heraldic design can then be followed to Persia, Syria, later to Byzantium, and anyone who lived before the First World War will remember the double-headed eagle in the coats-of-arms of Czarist Russia and the Austro-Hungarian monarchy.

Look now at this Sumerian picture (Figure 4). The two eagle-headed men are nearly but not quite symmetric; why not? In plane geometry reflection in a vertical line l can also be brought about by rotating the plane in space around the axis l by 180°. If you look at their arms you would say these two monsters arise from each other by such rotation; the overlappings depicting their position in space prevent the plane picture from having bilaterial symmetry. Yet the artist aimed at that symmetry by giving both figures a half turn toward the observer and also by the arrangement of feet and wings: the drooping wing is the right one in the left figure, the left one in the right figure.

The designs on the cylindrical Babylonian seal stones are frequently ruled by heraldic symmetry. I remember seeing in the collection of my former colleague, the late Ernst Herzfeld, samples where for symmetry's sake not the head, but the lower bull-shaped part of a god's body, rendered in profile, was doubled and given four instead of two hind legs. In Christian times one may see an analogy in certain representations of the Eucharist as on this Byzantine platen (Figure 5), where two symmetric

Christs are facing the disciples. But here symmetry is not complete and has clearly more than formal significance, for Christ on one side breaks the bread, on the other pours the wine.

Between Sumeria and Byzantium let me insert Persia: These enameled sphinxes (Figure 6) are from Darius' palace in Susa built in the days of Marathon. Crossing the Aegean we find these floor patterns (Figure 7) at the Megaron in Tiryns, late helladic about 1200 B.C. Who believes strongly in historic continuity and dependence will trace the graceful designs of marine life, dolphin and octopus, back to the Minoan culture of Crete, the heraldic symmetry to oriental, in the last instance Sumerian, influence. Skipping thousands of years we still see the same influences at work in this plaque (Figure 8) from the altar enclosure in the dome of Torcello, Italy, eleventh century A.D. The peacocks drinking from a pine well among vine leaves are an ancient Christian symbol of immortality, the structural heraldic symmetry is oriental.

For in contrast to the orient, occidental art, like life itself, is inclined to mitigate, to loosen, to modify, even to break strict symmetry. But seldom is asymmetry merely the absence of symmetry. Even in asymmetric designs one feels symmetry as the norm from which one deviates under the influence of forces of non-formal character. I think the riders

FIGURE 5

FIGURE 6

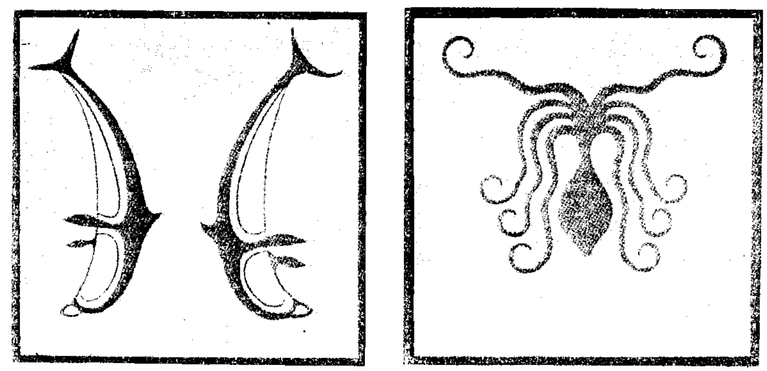

FIGURE 7

from the famous Etruscan Tomb of the Triclinium at Corneto (Figure 9) provide a good example. I have already mentioned representations of the Eucharist with Christ duplicated handing out bread and wine. The central group, Mary flanked by two angels, in this mosaic of the Lord's Ascension (Figure 12) in the cathedral at Monreale, Sicily (twelfth century), has almost perfect symmetry. [The band ornaments above and below the mosaic will demand our attention in the second lecture.] The principle of symmetry is somewhat less strictly observed in an earlier mosaic from San Apollinare in Ravenna (Figure 10), showing Christ surrounded by an angelic guard of honor. For instance Mary in the Monreale mosaic raises both hands symmetrically, in the *orans* gesture; here only the right hands

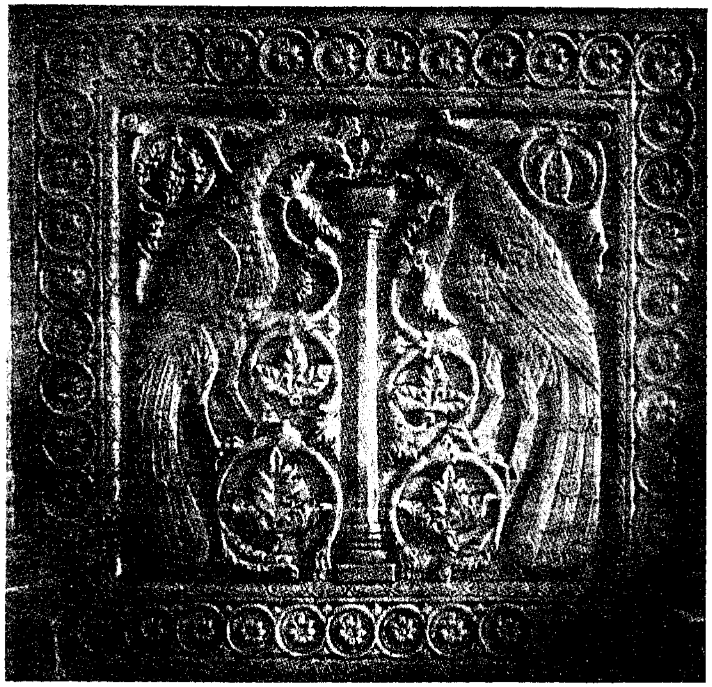

FIGURE 8

are raised. Asymmetry has made further inroads in the next picture (Figure 11), a Byzantine relief ikon from San Marco, Venice. It is a Deësis, and, of course, the two figures praying for mercy as the Lord is about to pronounce the last judgment cannot be mirror images of each other; for to the right stands his Virgin Mother, to the left John the Baptist. You may also think of Mary and John the Evangelist on both sides of the cross in crucifixions as examples of broken symmetry.

Clearly we touch ground here where the precise geometric notion of bilateral symmetry begins to dissolve into the vague notion of *Ausgewogenheit*, balanced design with which we started. "Symmetry," says Dagobert Frey in an article *On the Problem of Symmetry in Art*,[4] "signifies rest and binding, asymmetry motion and loosening, the one order and law, the other arbitrariness and accident, the one formal rigidity and constraint, the other life, play and freedom." Wherever God or Christ are represented as symbols for everlasting truth or justice they are given in the symmetric frontal view, not in profile. Probably for similar reasons public buildings and houses of worship, whether they are Greek temples or Christian basilicas and cathedrals, are bilaterally symmetric. It is, how-

[4] Studium Generale, p. 276.

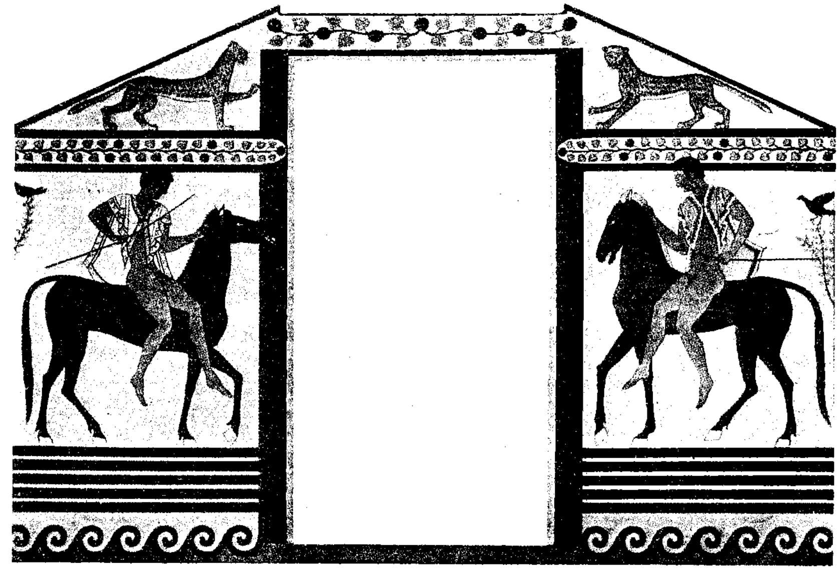

FIGURE 9

ever, true that not infrequently the two towers of Gothic cathedrals are different, as for instance in Chartres. But in practically every case this seems to be due to the history of the cathedral, namely to the fact that the towers were built in different periods. It is understandable that a later time was no longer satisfied with the design of an earlier period; hence one may speak here of historic asymmetry. Mirror images occur where there is a mirror, be it a lake reflecting a landscape or a glass mirror into which a woman looks. Nature as well as painters make use of this motif. I trust, examples will easily come to your mind. The one most familiar to me, because I look at it in my study every day, is Hodler's *Lake of Silvaplana*.

While we are about to turn from art to nature, let us tarry a few minutes and first consider what one may call the *mathematical philosophy of left and right*. To the scientific mind there is no inner difference, no polarity between left and right, as there is for instance in the contrast of male and female, or of the anterior and posterior ends of an animal. It requires an arbitrary act of choice to determine what is left and what is right. But after it is made for one body it is determined for every body. I must try to make this a little clearer. In space the distinction of left and right concerns the orientation of a screw. If you speak of turning left you mean that the sense in which you turn combined with the upward direction from foot to head of your body forms a left screw. The daily rotation of the earth together with the direction of its axis from South to North Pole is

FIGURE 10

FIGURE 11

a left screw, it is a right screw if you give the axis the opposite direction. There are certain crystalline substances called optically active which betray the inner asymmetry of their constitution by turning the polarization

plane of polarized light sent through them either to the left or to the right; by this, of course, we mean that the sense in which the plane rotates while the light travels in a definite direction, combined with that direction, forms a left screw (or a right one, as the case may be). Hence when we said above and now repeat in a terminology due to Leibniz, that left and right are *indiscernible*, we want to express that the inner structure of space does not permit us, except by arbitrary choice, to distinguish a left from a right screw.

I wish to make this fundamental notion still more precise, for on it depends the entire theory of relativity, which is but another aspect of symmetry. According to Euclid one can describe the structure of space by a number of basic relations between points, such as ABC lie on a straight line, $ABCD$ lie in a plane, AB is congruent CD. Perhaps the best way of describing the structure of space is the one Helmholtz adopted: by the

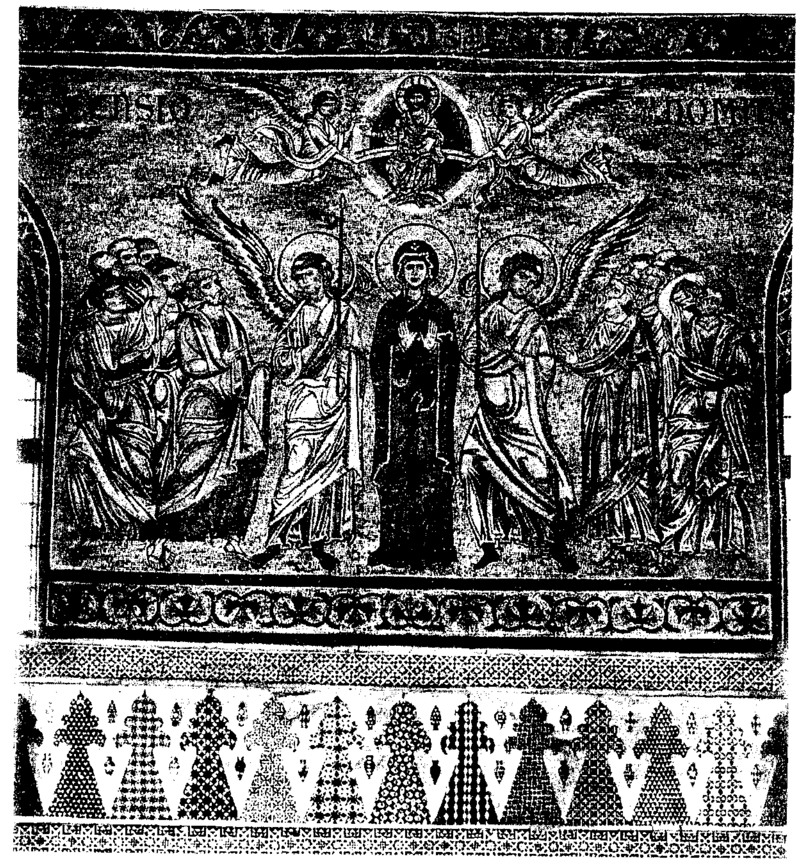

FIGURE 12

single notion of *congruence* of figures. A mapping S of space associates with every point p a point $p' : p \to p'$. A pair of mappings S, $S' : p \to p'$, $p' \to p$, of which the one is the inverse of the other, so that if S carries p into p' then S' carries p' back into p and vice versa, is spoken of as a pair

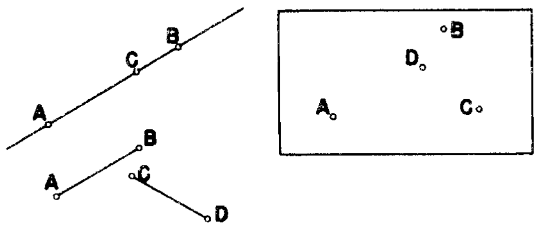

FIGURE 13

of one-to-one mappings or *transformations*. A transformation which preserves the structure of space—and if we define this structure in the Helmholtz way, that would mean that it carries any two congruent figures into two congruent ones—is called an *automorphism* by the mathematicians. Leibniz recognized that this is the idea underlying the geometric concept of similarity. An automorphism carries a figure into one that in Leibniz' words is "indiscernible from it if each of the two figures is considered by itself." What we mean then by stating that left and right are of the same essence is the fact that *reflection in a plane is an automorphism*.

Space as such is studied by geometry. But space is also the medium of all physical occurrences. The structure of the physical world is revealed by the general laws of nature. They are formulated in terms of certain basic quantities which are functions in space and time. We would conclude that the physical structure of space "contains a screw," to use a

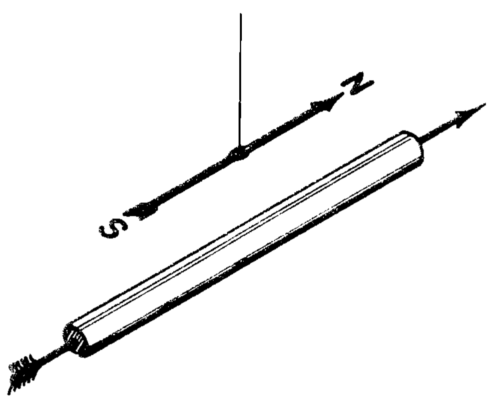

FIGURE 14

suggestive figure of speech, if these laws were not invariant throughout with respect to reflection. Ernst Mach tells of the intellectual shock he received when he learned as a boy that a magnetic needle is deflected in a certain sense, to the left or to the right, if suspended parallel to a wire through which an electric current is sent in a definite direction (Figure 14). Since the whole geometric and physical configuration, including the electric current and the south and north poles of the magnetic needle, to all appearances, are symmetric with respect to the plane E laid through the wire and the needle, the needle should react like Buridan's ass between equal bundles of hay and refuse to decide between left and right, just as scales of equal arms with equal weights neither go down on their left nor on their right side but stay horizontal. But appearances are sometimes deceptive. Young Mach's dilemma was the result of a too hasty assumption concerning the effect of reflection in E on the electric current and the positive and negative magnetic poles of the needle: while we know a priori how geometric entities fare under reflection, we have to learn from nature how the physical quantities behave. And this is what we find: under reflection in the plane E the electric current preserves its direction, but the magnetic south and north poles are interchanged. Of course this way out, which re-establishes the equivalence of left and right, is possible only because of the essential equality of positive and negative magnetism. All doubts were dispelled when one found that the magnetism of the needle has its origin in molecular electric currents circulating around the needle's direction; it is clear that under reflection in the plane E such currents change the sense in which they flow.

The net result is that in all physics nothing has shown up indicating an intrinsic difference of left and right. Just as all points and all directions in space are equivalent, so are left and right. Position, direction, left and right are *relative* concepts. In language tinged with theology this issue of relativity was discussed at great length in a famous controversy between Leibniz and Clarke, the latter a clergyman acting as the spokesman for Newton.[5] Newton with his belief in absolute space and time considers motion a proof of the creation of the world out of God's arbitrary will, for otherwise it would be inexplicable why matter moves in this rather than in any other direction. Leibniz is loath to burden God with such decisions as lack "sufficient reason." Says he, "Under the assumption that space be something in itself it is impossible to give a reason why God should have put the bodies (without tampering with their mutual distances and relative positions) just at this particular place and not somewhere else; for instance, why He should not have arranged everything in the opposite order by turning East and West about. If, on the other hand,

[5] See G. W. Leibniz, *Philosophische Schriften*, ed. Gerhardt (Berlin 1875 seq.), VII, pp. 352–440, in particular Leibniz' third letter, §5.

space is nothing more than the spatial order and relation of things then the two states supposed above, the actual one and its transposition, are in no way different from each other . . . and therefore it is a quite inadmissible question to ask why one state was preferred to the other." By pondering the problem of left and right Kant was first led to his conception of space and time as forms of intuition.[6] Kant's opinion seems to have been this: If the first creative act of God had been the forming of a left hand then this hand, even at the time when it could be compared to nothing else, had the distinctive character of left, which can only intuitively but never conceptually be apprehended. Leibniz contradicts: According to him it would have made no difference if God had created a "right" hand first rather than a "left" one. One must follow the world's creation a step further before a difference can appear. Had God, rather than making first a left and then a right hand, started with a right hand and then formed another right hand, He would have changed the plan of the universe not in the first but in the second act, by bringing forth a hand which was equally rather than oppositely oriented to the first-created specimen.

Scientific thinking sides with Leibniz. Mythical thinking has always taken the contrary view as is evinced by its usage of right and left as symbols for such polar opposites as good and evil. You need only think of the double meaning of the word *right* itself. In this detail from Michelangelo's famous *Creation of Adam* from the Sistine Ceiling (Figure 15) God's right hand, on the right, touches life into Adam's left.

People shake right hands. *Sinister* is the Latin word for left, and heraldry still speaks of the left side of the shield as its sinister side. But *sinistrum* is at the same time that which is evil, and in common English only this figurative meaning of the Latin word survives.[7] Of the two malefactors who were crucified with Christ, the one who goes with Him to paradise is on His right. St. Matthew, Chapter 25, describes the last judgment as follows: "And he shall set the sheep on his right hand but the goats on the left. Then shall the King say unto them on his right hand, Come ye, blessed of my Father, inherit the Kingdom prepared for you from the foundation of the world. . . . Then he shall say also unto them on the left hand, Depart from me, ye cursed, into everlasting fire, prepared for the devil and his angels."

I remember a lecture Heinrich Wölfflin once delivered in Zurich on "Right and left in paintings"; together with an article on "The problem of inversion (Umkehrung) in Raphael's tapestry cartoons," you now find it printed in abbreviated form in his *Gedanken zur Kunstgeschichte*, 1941.

[6] Besides his "Kritik der reinen Vernunft" see especially § 13 of the *Prolegomena zu einer jeden künftigen Metaphysik*. . . .

[7] I am not unaware of the strange fact that as a *terminus technicus* in the language of the Roman augurs *sinistrum* had just the opposite meaning of propitious.

FIGURE 15

By a number of examples, as Raphael's *Sistine Madonna* and Rembrandt's etching *Landscape with the three trees*, Wölfflin tries to show that right in painting has another *Stimmungswert* than left. Practically all methods of reproduction interchange left and right, and it seems that former times were much less sensitive than we are toward such inversion. (Even Rembrandt did not hesitate to bring his Descent from the Cross as a converse etching upon the market.) Considering that we do a lot more reading than the people, say, of the sixteenth century, this suggests the hypothesis that the difference pointed out by Wöfflin is connected with our habit of reading from left to right. As far as I remember, he himself rejected this as well as a number of other psychological explanations put forward in the discussion after his lecture. The printed text concludes with the remark that the problem "obviously has deep roots, roots which reach down to the very foundations of our sensuous nature." I for my part am disinclined to take the matter that seriously.[8]

In science the belief in the equivalence of left and right has been upheld even in the face of certain biological facts presently to be mentioned which seem to suggest their inequivalence even more strongly than does the deviation of the magnetic needle which shocked young Mach. The same

[8] Cf. also A. Faistauer, "Links und rechts im Bilde," Amicis, *Jahrbuch der österreichischen Galerie*, 1926, p. 77; Julius v. Schlosser, "Intorno alla lettura dei quadri," *Critica* 28, 1930, p. 72; Paul Oppé, "Right and left in Raphael's cartoons," *Journal of the Warburg and Courtauld Institutes* 7, 1944, p. 82.

problem of equivalence arises with respect to *past and future*, which are interchanged by inverting the direction of time, and with respect to *positive and negative electricity*. In these cases, especially in the second, it is perhaps clearer than for the pair left-right that a priori evidence is not sufficient to settle the question; the empirical facts have to be consulted. To be sure, the role which past and future play in our consciousness would indicate their intrinsic difference—the past knowable and unchangeable, the future unknown and still alterable by decisions taken now—and one would expect that this difference has its basis in the physical laws of nature. But those laws of which we can boast a reasonably certain knowledge are invariant with respect to the inversion of time as they are with respect to the interchange of left and right. Leibniz made it clear that the temporal modi past and future refer to the *causal structure* of the world. Even if it is true that the exact "wave laws" formulated by quantum physics are not altered by letting time flow backward, the metaphysical idea of causation, and with it the one way character of time, may enter physics through the statistical interpretation of those laws in terms of probability and particles. Our present physical knowledge leaves us even more uncertain about the equivalence or non-equivalence of positive and negative electricity. It seems difficult to devise physical laws in which they are not intrinsically alike; but the negative counterpart of the positively charged proton still remains to be discovered.

This half-philosophical excursion was needed as a background for the discussion of the left-right symmetry in nature; we had to understand that the general organization of nature possesses that symmetry. But one will not expect that any special object of nature shows it to perfection. Even so, it is surprising to what extent it prevails. There must be a reason for this, and it is not far to seek: a state of equilibrium is likely to be symmetric. More precisely, under conditions which determine a unique state of equilibrium the symmetry of the conditions must carry over to the state of equilibrium. Therefore tennis balls and stars are spheres; the earth would be a sphere too if it did not rotate around an axis. The rotation flattens it at the poles but the rotational or cylindrical symmetry around its axis is preserved. The feature that needs explanation is, therefore, not the rotational symmetry of its shape but the deviations from this symmetry as exhibited by the irregular distribution of land and water and by the minute crinkles of mountains on its surface. It is for such reasons that in his monograph on the left-right problem in zoology Wilhelm Ludwig says hardly a word about the origin of the bilateral symmetry prevailing in the animal kingdom from the echinoderms upward, but in great detail discusses all sorts of secondary asymmetries superimposed upon the symmetrical ground plan.[9] I quote: "The human body like that of the

[9] W. Ludwig, *Rechts-links-Problem im Tierreich und beim Menschen*, Berlin 1932.

other vertebrates is basically built bilateral-symmetrically. All asymmetries occurring are of secondary character, and the more important ones affecting the inner organs are chiefly conditioned by the necessity for the intestinal tube to increase its surface out of proportion to the growth of the body, which lengthening led to an asymmetric folding and rolling-up. And in the course of phylogenetic evolution these first asymmetries concerning the intestinal system with its appendant organs brought about asymmetries in other organ systems." It is well known that the heart of mammals is an asymmetric screw, as shown by the schematic drawing of Figure 16.

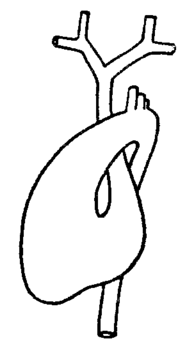

FIGURE 16

If nature were all lawfulness then every phenomenon would share the full symmetry of the universal laws of nature as formulated by the theory of relativity. The mere fact that this is not so proves that *contingency* is an essential feature of the world. Clarke in his controversy with Leibniz admitted the latter's principle of sufficient reason but added that the sufficient reason often lies in the mere will of God. I think, here Leibniz the rationalist is definitely wrong and Clarke on the right track. But it would have been more sincere to deny the principle of sufficient reason altogether instead of making God responsible for all that is unreason in the world. On the other hand Leibniz was right against Newton and Clarke with his insight into the principle of relativity. The truth as we see it today is this: The laws of nature do not determine uniquely the one world that actually exists, not even if one concedes that two worlds arising from each other by an automorphic transformation, i.e., by a transformation which preserves the universal laws of nature, are to be considered the same world.

If for a lump of matter the overall symmetry inherent in the laws of nature is limited by nothing but the accident of its position P then it will assume the form of a sphere around the center P. Thus the lowest forms of animals, small creatures suspended in water, are more or less spherical. For forms fixed to the bottom of the ocean the direction of gravity is an important factor, narrowing the set of symmetry operations from all rotations around the center P to all rotations about an axis. But for animals capable of self-motion in water, air, or on land both the postero-anterior direction in which their body moves and the direction of gravity are of decisive influence. After determination of the antero-posterior, the dorso-ventral, and thereby of the left-right axes, only the distinction between left and right remains arbitrary, and at this stage no higher symmetry than the bilateral type can be expected. Factors in the phylogenetic evolution that tend to introduce inheritable differences between left and right are likely to be held in check by the advantage an animal derives from the bilateral formation of its organs of motion, cilia or muscles and limbs: in case of their asymmetric development a screw-wise instead of a straightforward motion would naturally result. This may help to explain why our limbs obey the law of symmetry more strictly than our inner organs. Aristophanes in Plato's *Symposium* tells a different story of how the transition from spherical to bilateral symmetry came about. Originally, he says, man was round, his back and sides forming a circle. To humble their pride and might Zeus cut them into two and had Apollo turn their faces and genitals around; and Zeus had threatened, "If they continue insolent I will split them again and they shall hop around on a single leg."

The most striking examples of symmetry in the inorganic world are the crystals. The gaseous and the crystalline are two clear-cut states of matter which physics finds relatively easy to explain; the states in between these two extremes, like the fluid and the plastic states, are somewhat less amenable to theory. In the gaseous state molecules move freely around in space with mutually independent random positions and velocities. In the crystalline state atoms oscillate about positions of equilibrium as if they were tied to them by elastic strings. These positions of equilibrium form a fixed regular configuration in space.[10]. . . While most of the thirty-two geometrically possible systems of crystal symmetry involve bilateral symmetry, not all of them do. Where it is not involved we have the possibility of so-called enantiomorph crystals which exist in a laevo- and dextro-form, each form being a mirror image of the other, like left and right hands. A substance which is optically active, i.e., turns the plane of polarized light either left or right, can be expected to crystallize in such asymmetric forms. If the laevo-form exists in nature one would assume that the dextro-form exists likewise, and that in the average both occur with equal frequencies. In 1848 Pasteur made the discovery that when the sodium ammonium salt of optically inactive racemic acid was recrystallized from an aqueous solution at a lower temperature the deposit consisted of two kinds of tiny crystals which were mirror images of each other. They were carefully separated, and the acids set free from the one and the other proved to

[10] [In a later lecture Weyl explains how the visible symmetry of crystals derives from their regular atomic arrangement. ED.]

have the same chemical composition as the racemic acid, but one was optically laevo-active, the other dextro-active. The latter was found to be identical with the tartaric acid present in fermenting grapes, the other had never before been observed in nature. "Seldom," says F. M. Jaeger in his lectures *On the principle of symmetry and its applications in natural science*, "has a scientific discovery had such far-reaching consequences as this one had."

Quite obviously some accidents hard to control decide whether at a spot of the solution a laevo- or dextro-crystal comes into being; and thus in agreement with the symmetric and optically inactive character of the solution as a whole and with the law of chance the amounts of substance deposited in the one and the other form at any moment of the process of crystallization are equal or very nearly equal. On the other hand nature, in giving us the wonderful gift of grapes so much enjoyed by Noah, produced only one of the forms, and it remained for Pasteur to produce the other! This is strange indeed. It is a fact that most of the numerous carbonic compounds occur in nature in one, either the laevo- or the dextro-form only. The sense in which a snail's shell winds is an inheritable character founded in its genetic constitution, as is the "left heart" and the winding of the intestinal duct in the species *Homo sapiens*. This does not exclude that inversions occur, e.g. *situs inversus* of the intestines of man occurs with a frequency of about 0.02 per cent; we shall come back to that later! Also the deeper chemical constitution of our human body shows that we have a screw, a screw that is turning the same way in every one of us. Thus our body contains the dextro-rotatory form of glucose and laevo-rotatory form of fructose. A horrid manifestation of this genotypical asymmetry is a metabolic disease called phenylketonuria, leading to insanity, that man contracts when a small quantity of laevo-phenylalanine is added to his food, while the dextro-form has no such disastrous effects. To the asymmetric chemical constitution of living organisms one must attribute the success of Pasteur's method of isolating the laevo- and dextro-forms of substances by means of the enzymatic action of bacteria, moulds, yeasts, and the like. Thus he found that an originally inactive solution of some racemate became gradually laevo-rotatory if *Penicillium glaucum* was grown in it. Clearly the organism selected for its nutriment that form of the tartaric acid molecule which best suited its own asymmetric chemical constitution. The image of lock and key has been used to illustrate this specificity of the action of organisms.

In view of the facts mentioned and in view of the failure of all attempts to "activate" by mere chemical means optically inactive material,[11] it is understandable that Pasteur clung to the opinion that the production

[11] There is known today one clear instance, the reaction of nitrocinnaminacid with bromine where circular-polarized light generates an optically active substance.

of single optically active compounds was the very prerogative of life. In 1860 he wrote, "This is perhaps the only well-marked line of demarkation that can at present be drawn between the chemistry of dead and living matter." Pasteur tried to explain his very first experiment where racemic acid was transformed by recrystallization into a mixture of laevo- and dextro-tartaric acid by the action of bacteria in the atmosphere on his neutral solution. It is quite certain today that he was wrong; the sober physical explanation lies in the fact that at lower temperature a mixture of the two oppositely active tartaric forms is more stable than the inactive racemic form. If there is a difference in principle between life and death it does not lie in the chemistry of the material substratum; this has been fairly certain ever since Wöhler in 1828 synthesized urea from purely mineral material. But even as late as 1898 F. R. Japp in a famous lecture on "Stereochemistry and Vitalism" before the British Association upheld Pasteur's view in the modified form: "Only the living organisms, or the living intelligence with its conception of symmetry can produce this result (i.e. asymmetric compounds)." Does he really mean that it is Pasteur's intelligence that, by devising the experiment but to its own great surprise, *creates* the dual tartaric crystals? Japp continues, "Only asymmetry can beget asymmetry." The truth of that statement I am willing to admit; but it is of little help since there is no symmetry in the accidental past and present set-up of the actual world which begets the future.

There is however a real difficulty: Why should nature produce only one of the doublets of so many enantiomorphic forms the origin of which most certainly lies in living organisms? Pascual Jordan points to this fact as a support for his opinion that the beginnings of life are not due to chance events which, once a certain stage of evolution is reached, are apt to occur continuously now here now there, but rather to an event of quite singular and improbable character, occurring once by accident and then starting an avalanche by autocatalytic multiplication. Indeed had the asymmetric protein molecules found in plants and animals an independent origin in many places at many times, then their laevo- and dextro-varieties should show nearly the same abundance. Thus it looks as if there is some truth in the story of Adam and Eve, if not for the origin of mankind then for that of the primordial forms of life. It was in reference to these biological facts when I said before that if taken at their face value they suggest an intrinsic difference between left and right, at least as far as the constitution of the organic world is concerned. But we may be sure the answer to our riddle does not lie in any universal biological laws but in the accidents of the genesis of the organismic world. Pascual Jordan shows one way out; one would like to find a less radical one, for instance by reducing the asymmetry of the inhabitants on earth to some inherent, though accidental, asymmetry of the earth itself, or of the light received

on earth from the sun. But neither the earth's rotation nor the combined magnetic fields of earth and sun are of immediate help in this regard. Another possibility would be to assume that development actually started from an equal distribution of the enantiomorph forms, but that this is an unstable equilibrium which under a slight chance disturbance tumbled over.

From the phylogenetic problems of left and right let us finally turn to their ontogenesis. Two questions arise: Does the first division of the fertilized egg of an animal into two cells fix the median plane, so that one of the cells contains the potencies for its left, the other for its right half? Secondly what determines the plane of the first division? I begin with the second question. The egg of any animal above the protozoa possesses from the beginning a polar axis connecting what develops into the animal and the vegetative poles of the blastula. This axis together with the point where the fertilizing spermatozoon enters the egg determines a plane, and it would be quite natural to assume that this is the plane of the first division. And indeed there is evidence that it is so in many cases. Present opinion seems to incline toward the assumption that the primary polarity as well as the subsequent bilateral symmetry come about by external factors actualizing potentialities inherent in the genetic constitution. In many instances the direction of the polar axis is obviously determined by the attachment of the oozyte to the wall of the ovary, and the point of entrance of the fertilizing sperm is, as we said, at least one, and often the most decisive, of the determining factors for the median plane. But other agencies may also be responsible for the fixation of the one and the other. In the sea-weed *Fucus* light or electric fields or chemical gradients determine the polar axis, and in some insects and cephalopods the median plane appears to be fixed by ovarian influences before fertilization.[12] The underlying constitution on which these agencies work is sought by some biologists in an intimate preformed structure, of which we do not yet have a clear picture. Thus Conklin has spoken of a spongioplasmic framework, others of a cytoskeleton, and as there is now a strong tendency among biochemists to reduce structural properties to fibers, so much so that Joseph Needham in his Terry Lectures on *Order and life* (1936) dares the aphorism that biology is largely the study of fibers, one may expect them to find that that intimate structure of the egg consists of a framework of elongated protein molecules or fluid crystals.

[12] Julian S. Huxley and G. R. de Beer in their classical *Elements of embryology* (Cambridge University Press, 1934) give this formulation (Chapter XIV, Summary, p. 438): "In the earliest stages, the egg acquires a unitary organization of the gradient-field type in which quantitative differentials of one or more kinds extend across the substance of the egg in one or more directions. The constitution of the egg predetermines it to be able to produce a gradient-field of a particular type; however, the localization of the gradients is not predetermined, but is brought about by agencies external to the egg."

We know a little more about our first question whether the first mitosis of the cell divides it into left and right. Because of the fundamental character of bilateral symmetry the hypothesis that this is so seems plausible enough. However, the answer cannot be an unqualified affirmation. Even if the hypothesis should be true for the normal development we know from experiments first performed by Hans Driesch on the sea urchin that a single blastomere isolated from its partner in the two-cell stage develops into a whole gastrula differing from the normal one only by its smaller size. Here are Driesch's famous pictures. It must be admitted that this is not so for all species. Driesch's discovery led to the distinction between the actual and the potential destiny of the several parts of an egg. Driesch himself speaks of prospective significance (prospektive Bedeutung), as against prospective potency (prospektive Potenz); the latter is wider than the former, but shrinks in the course of development. Let me illustrate this basic point by another example taken from the determination of limb-buds of amphibia. According to experiments performed by R. G. Harrison, who transplanted discs of the outer wall of the body representing the buds of

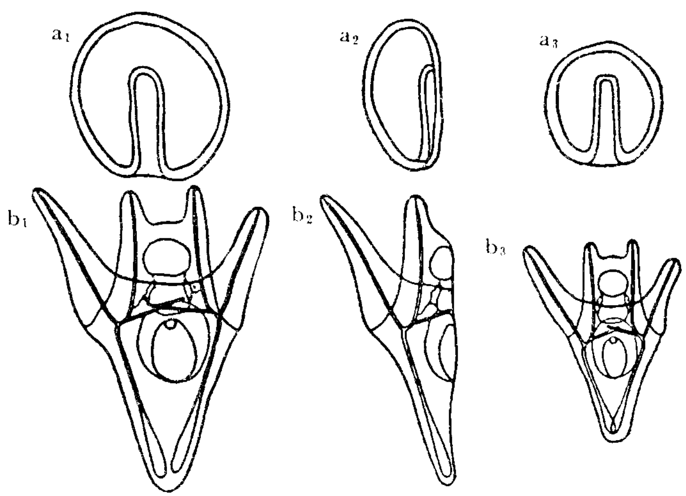

FIGURE 17

Experiments on pluripotence in "Echinus."
a_1 *and* b_1. *Normal gastrula and normal pluteus.*
a_2 *and* b_2. *Half-gastrula and half-pluteus, expected by Driesch.*
a_3 *and* b_3. *The small but whole gastrula and pluteus, which he actually obtained.*

future limbs, the antero-posterior axis is determined at a time when transplantation may still invert the dorso-ventral and the medio-lateral axes; thus at this stage the opposites of left and right still belong to the prospec-

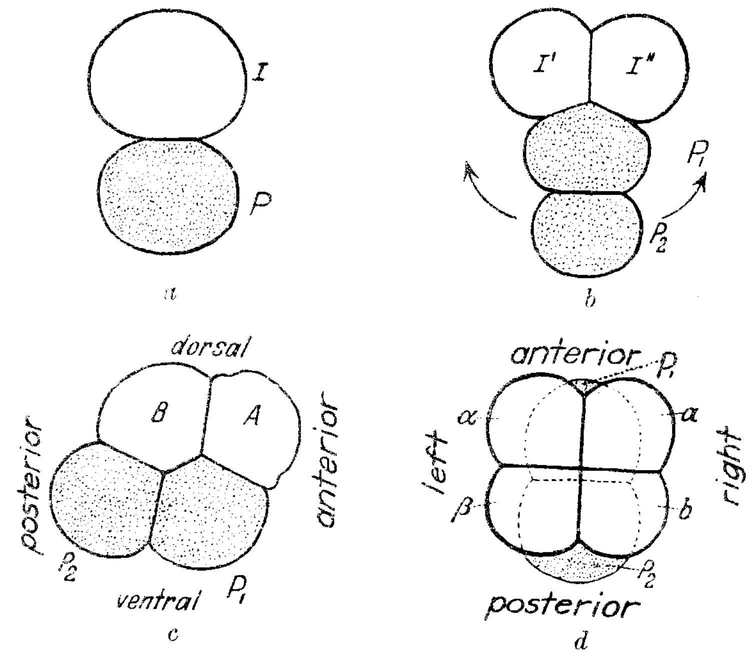

FIGURE 18

tive potencies of the discs, and it depends on the influence of the surrounding tissues in which way this potency will be actualized.

Driesch's violent encroachment on the normal development proves that the first cell division may not fix left and right of the growing organism for good. But even in normal development the plane of the first division may not be the median. The first stages of cell division have been closely studied for the worm *Ascaris megalocephala*, parts of whose nervous system are asymmetric. First the fertilized egg splits into a cell I and a smaller P of obviously different nature (Figure 18). In the next stage they divide along two perpendicular planes into $I' + I''$ and $P_1 + P_2$ respectively. Thereafter the handle $P_1 + P_2$ turns about so that P_2 comes into contact with either I' or I''; call the one it contacts B, the other A. We now have a sort of rhomboid and roughly AP_2 is the antero-posterior axis and BP_1 the dorsal-ventral one. Only the next division which along a plane perpendicular to the one separating A and B splits A as well as B into symmetric halves $A = a + \alpha$, $B = b + \beta$, is that which determines left and right. A further slight shift of the configuration destroys this bilateral symmetry. The question arises whether the direction of the two consecutive shifts is a chance event which decides first between anterior and posterior and then between left and right, or whether the constitution of the egg in its one-cell stage contains specific agents which determine the direction of these shifts. The hypothesis of the mosaic egg favoring the second hypothesis seems more likely for the species *Ascaris*.

There are known a number of cases of genotypical inversion where the genetic constitutions of two species are in the same relation as the atomic constitutions of two enantiomorph crystals. More frequent, however, is phenotypical inversion. Left-handedness in man is an example. I give another more interesting one. Several crustacea of the lobster type have two morphologically and functionally different claws, a bigger A and a smaller a. Assume that in normally developed individuals of our species, A is the right claw. If in a young animal you cut off the right claw, inversive regeneration takes place: the left claw develops into the bigger form A while at the place of the right claw a small one of type a is regenerated. One has to infer from such and similar experiences the bipotentiality of plasma, namely that all generative tissues which contain the potency of an asymmetric character have the potency of bringing forth both forms, so however that in normal development always one form develops, the left or the right. Which one is genetically determined, but abnormal external circumstances may cause inversion. On the basis of the strange phenomenon of inversive regeneration Wilhelm Ludwig developed the hypothesis that the decisive factors in asymmetry may not be such specific potencies as, say, the development of a "right claw of type A," but two R and L (right and left) agents which are distributed in the organism with a certain gradient, the concentration of one falling off from right to left, the other in the opposite direction. The essential point is that there is not one but that there are two opposite gradient fields R and L. Which is produced in greater strength is determined by the genetic constitution. If, however, by some damage to the prevalent agent the other previously suppressed one becomes prevalent, then inversion takes place. Being a mathematician and not a biologist I report with the utmost caution on these matters, which seem to me of highly hypothetical nature. But it is clear that the contrast of left and right is connected with the deepest problems concerning the phylogenesis as well as the ontogenesis of organisms.

TRANSLATORY, ROTATIONAL, AND RELATED SYMMETRIES

From bilateral, we shall now turn to other kinds of geometric symmetry. Even in discussing the bilateral type I could not help drawing in now and then such other symmetries as the cylindrical or the spherical ones. It seems best to fix the underlying general concept with some precision beforehand, and to that end a little mathematics is needed, for which I ask your patience. I have spoken of transformations. A mapping S of space associates with every space point p a point p' as its image. A special such mapping is the identity I carrying every point p into itself. Given two

mappings S, T, one can perform one after the other: if S carries p into p' and T carries p' into p'' then the resulting mapping, which we denote by ST, carries p into p''. A mapping may have an inverse S' such that $SS' = I$ and $S'S = I$; in other words, if S carries the arbitrary point p into p' then S' carries p' back into p, and a similar condition prevails with S' performed in the first and S in the second place. For such a one-to-one mapping S the word transformation was used in the first lecture; let the inverse be denoted by S^{-1}. Of course, the identity I is a transformation, and I itself is its inverse. Reflection in a plane, the basic operation of bilateral symmetry, is such that its iteration SS results in the identity; in other words, it is its own inverse. In general composition of mappings is not commutative; ST need not be the same as TS. Take for instance a point o in a plane and let S be a horizontal translation carrying o into o_1 and T a rotation around o by $90°$. Then ST carries o into the point o_2 (Figure 19), but TS carries o into o_1. If S is a transformation with the inverse S^{-1}, then S^{-1} is also a transformation and its inverse is S. The composite of two transformations ST is a transformation again, and $(ST)^{-1}$ equals $T^{-1}S^{-1}$ (in this order!). With this rule, although perhaps not with its mathematical expression, you are all familiar. When you dress, it is not immaterial in which order you perform the operations; and when in dressing you start with the shirt and end up with the coat, then in undressing you observe the opposite order; first take off the coat and the shirt comes last.

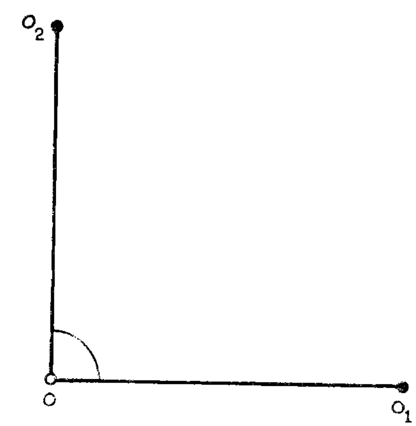

FIGURE 19

I have further spoken of a special kind of transformations of space called similarity by the geometers. But I preferred the name of automorphisms for them, defining them with Leibniz as those transformations which leave the structure of space unchanged. For the moment it is immaterial wherein that structure consists. From the very definition it is clear that the identity I is an automorphism, and if S is, so is the inverse S^{-1}. Moreover the composite ST of two automorphisms S, T is again an automorphism. This is only another way of saying that (1) every figure is similar to itself, (2) if figure F' is similar to F then F is similar to F', and (3) if F is similar to F' and F' to F'' then F is similar to F''. The mathematicians have adopted the word *group* to describe this situation and therefore say that the *automorphisms form a group*. Any totality, any set Γ of transformations form a group provided the following conditions are satisfied: (1) the identity I belongs to Γ; (2) if S belongs to Γ then its inverse S^{-1} does; (3) if S and T belong to Γ then the composite ST does.

One way of describing the structure of space, preferred by both Newton and Helmholtz, is through the notion of congruence. Congruent parts of space V, V' are such as can be occupied by the same rigid body in two of its positions. If you move the body from the one into the other position the particle of the body covering a point p of V will afterwards cover a certain point p' of V', and thus the result of the motion is a mapping $p \to p'$ of V upon V'. We can extend the rigid body either actually or in imagination so as to cover an arbitrarily given point p of space, and hence the congruent mapping $p \to p'$ can be extended to the entire space. Any such congruent transformation—I call it by that name because it evidently has an inverse $p' \to p$—is a similarity or an automorphism; you can easily convince yourselves that this follows from the very concepts. It is evident moreover that the congruent transformations form a group, a subgroup of the group of automorphisms. In more detail the situation is this. Among the similarities there are those which do not change the dimensions of a body; we shall now call them congruences. A congruence is either proper, carrying a left screw into a left and a right one into a right, or it is improper or reflexive, changing a left screw into a right one and vice versa. The proper congruences are those transformations which a moment ago we called congruent transformations, connecting the positions of points of a rigid body before and after a motion. We shall now call them simply motions (in a nonkinematic geometric sense) and call the improper congruences reflections, after the most important example: reflection in a plane, by which a body goes over into its mirror image. Thus we have this step-wise arrangement: similarities $\to$ congruences $=$ similarities without change of scale $\to$ motions $=$ proper congruences. The congruences form a subgroup of the similarities, the motions form a subgroup of the group of congruences, of index 2. The latter addition means that if B is any given improper congruence, we obtain all improper congruences in the form BS by composing B with all possible proper congruences S. Hence the proper congruences form one half, and the improper ones another half, of the group of all congruences. But only the first half is a

group; for the composite AB of two improper congruences A, B is a proper congruence.

A congruence leaving the point O fixed may be called *rotation* around O; thus there are proper and improper rotations. The rotations around a

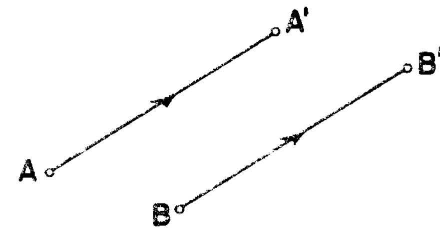

FIGURE 20

given center O form a group. The simplest type of congruences are the *translations*. A translation may be represented by a vector $\overrightarrow{AA'}$; for if a translation carries a point A into A' and the point B into B' then BB' has the same direction and length as AA', in other words the vector $\overrightarrow{BB'} = \overrightarrow{AA'}$.[13] The translations form a group; indeed the succession of the two translations $\overrightarrow{AB}$, $\overrightarrow{BC}$ results in the translation $\overrightarrow{AC}$.

What has all this to do with symmetry? It provides the adequate mathematical language to define it. Given a spatial configuration $\mathfrak{F}$, those automorphisms of space which leave $\mathfrak{F}$ unchanged form a group Γ, and *this group describes exactly the symmetry possessed by* $\mathfrak{F}$. Space itself has the full symmetry corresponding to the group of all automorphisms, of all similarities. The symmetry of any figure in space is described by a subgroup of that group. Take for instance the famous pentagram (Figure 21) by which Dr. Faust banned Mephistopheles the devil. It is carried into itself by the five proper rotations around its center O, the angles of which are multiples of $360°/5$ (including the identity), and then by the five reflections in the lines joining O with the five vertices. These ten operations form a group, and that group tells us what sort of symmetry the pentagram possesses. Hence the natural generalization which leads from bilateral symmetry to symmetry in this wider geometric sense consists in replacing reflection in a plane by any group of automorphisms. The circle in a plane with center

[13] While a segment has only length, a vector has length and direction. A vector is really the same thing as a translation, although one uses different phraseologies for vectors and translations. Instead of speaking of the translation $\mathfrak{a}$ which carries the point A into A' one speaks of the vector $\mathfrak{a} = \overrightarrow{AA'}$; and instead of the phrase: the translation $\mathfrak{a}$ carries A into A' one says that A' is the end point of the vector $\mathfrak{a}$ laid off from A. The same vector laid off from B ends in B' if the translation carrying A into A' carries B into B'.

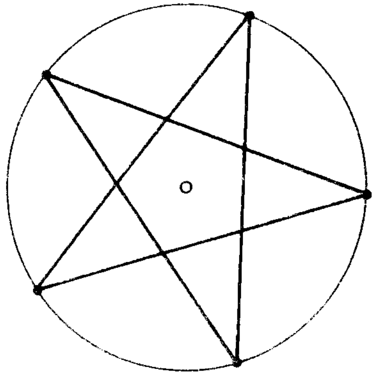

FIGURE 21

O and the sphere in space around O have the symmetry described by the group of all plane or spatial rotations respectively.

If a figure $\mathfrak{F}$ does not extend to infinity then an automorphism leaving the figure invariant must be scale-preserving and hence a congruence, unless the figure consists of one point only. Here is the simple proof. Had we an automorphism leaving $\mathfrak{F}$ unchanged, but changing the scale, then either this automorphism or its inverse would increase (and not decrease) all linear dimensions in a certain proportion $a : 1$ where a is a number greater than 1. Call that automorphism S, and let α, β be two different points of our figure $\mathfrak{F}$. They have a positive distance d. Iterate the transformation S,

$$S = S^1, \; SS = S^2, \; SSS = S^3, \; \ldots .$$

The n-times iterated transformation S^n carries α and β into two points α_n, β_n of our figure whose distance is $d \cdot a^n$. With increasing exponent n this distance tends to infinity. But if our figure $\mathfrak{F}$ is bounded, there is a number c such that no two points of $\mathfrak{F}$ have a distance greater than c. Hence a contradiction arises as soon as n becomes so large that $d \cdot a^n > c$. The argument shows another thing: Any finite group of automorphisms consists exclusively of congruences. For if it contains an S that enlarges linear dimensions at the ratio $a : 1$, $a > 1$, then all the infinitely many iterations $S^1, S^2, S^3, \cdots$ contained in the group would be different because they enlarge at different scales $a^1, a^2, a^3, \cdots$. For such reasons as these we shall almost exclusively consider groups of congruences—even if we have to do with actually or potentially infinite configurations such as band ornaments and the like.

After these general mathematical considerations let us now take up some special groups of symmetry which are important in art or nature. The operation which defines bilateral symmetry, mirror reflection, is essentially a one-dimensional operation. A straight line can be reflected in any of its points O; this reflection carries a point P into that point P' that has the same distance from O but lies on the other side. Such reflections are the only improper congruences of the one-dimensional line, whereas its only proper congruences are the translations. Reflection in O followed by the translation OA yields reflection in that point A_1 which halves the distance OA. A figure which is invariant under a translation t shows what in the art of ornament is called "infinite rapport," i.e. repetition in a regular spatial rhythm. A pattern invariant under the translation t is also invariant under its iterations $t^1, t^2, t^3, \cdots$, moreover under the identity $t^0 = I$, and under the inverse t^{-1} of t and its iterations $t^{-1}, t^{-2}, t^{-3}, \cdots$. If t shifts the line by the amount a then t^n shifts it by the amount

$$na \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

Hence if we characterize a translation t by the shift a it effects then the iteration or power t^n is characterized by the multiple na. All translations carrying into itself a given pattern of infinite rapport on a straight line are in this sense multiples na of one basic translation a. This rhythmic may be combined with reflexive symmetry. If so the centers of reflections follow each other at half the distance $\frac{1}{2}a$. Only these two types of symmetry, as illustrated by Figure 22, are possible for a one-dimensional pattern or "ornament." (The crosses $\times$ mark the centers of reflection.)

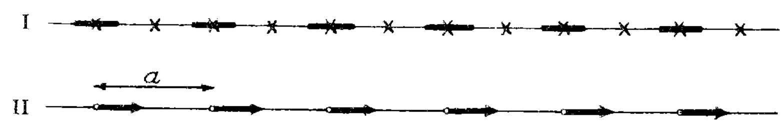

FIGURE 22

Of course the real band ornaments are not strictly one-dimensional, but their symmetry as far as we have described it now makes use of their longitudinal dimension only. Here are some simple examples from Greek art. The first (Figure 23) which shows a very frequent motif, the palmette, is of type I (translation + reflection). The next (Figure 24) are without reflections (type II). This frieze of Persian bowmen from Darius' palace in Susa (Figure 25) is pure translation; but you should notice that the basic translation covers twice the distance from man to man because the costumes of the bowmen alternate. Once more I shall point out the Monreale mosaic of the Lord's Ascension (Figure 10), but this time drawing your attention to the band ornaments framing it. The widest, carried out in a peculiar technique, later taken up by the Cosmati, displays the translatory symmetry only by repetition of the outer contour of the basic tree-like motif, while each copy is filled by a different highly

FIGURE 23

FIGURE 24

symmetric two-dimensional mosaic. The palace of the doges in Venice (Figure 26) may stand for translatory symmetry in architecture. Innumerable examples could be added.

As I said before, band ornaments really consist of a two-dimensional strip around a central line and thus have a second transversal dimension. As such they can have further symmetries. The pattern may be carried into itself by reflection in the central line l; let us distinguish this as longitudinal reflection from the transversal reflection in a line perpendicular to l. Or the pattern may be carried into itself by longitudinal reflection combined with the translation by $\frac{1}{2}a$ (longitudinal slip reflection). A fre-

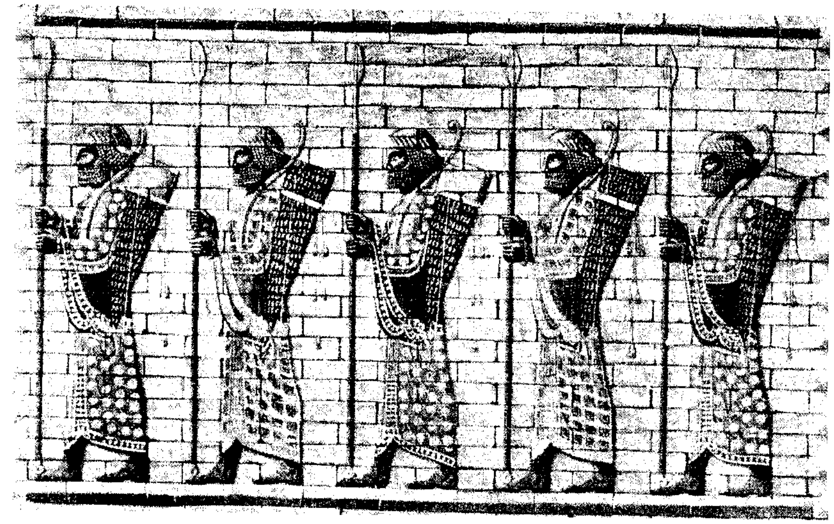

FIGURE 25

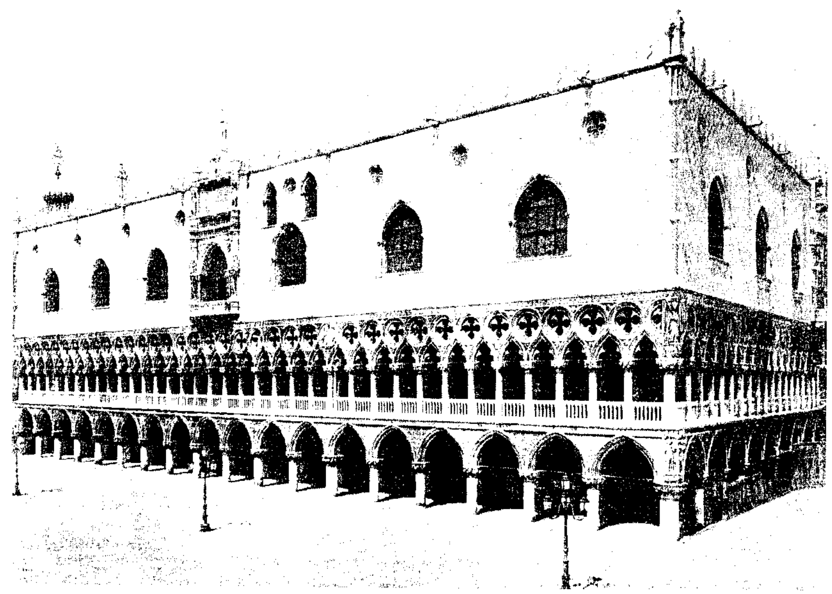

FIGURE 26

quent motif in band ornaments are cords, strings, or plaits of some sort, the design of which suggests that one strand crosses the other in space (and thus makes part of it invisible). If this interpretation is accepted, further operations become possible; for example, reflection in the plane of the ornament would change a strand slightly above the plane into one below. All this can be thoroughly analyzed in terms of group theory as is for instance done in a section of Andreas Speiser's book, *Theorie der Gruppen von endlicher Ordnung*.

In the organic world the translatory symmetry, which the zoologists call metamerism, is seldom as regular as bilateral symmetry frequently is. A maple shoot and a shoot of *Angraecum distichum* (Figure 27) may serve as examples.[14] In the latter case translation is accompanied by longitudinal slip reflection. Of course the pattern does not go on into infinity (nor does a band ornament), but one may say that it is potentially infinite at least in one direction, as in the course of time ever new segments separated from each other by a bud come into being. Goethe said of the tails of vertebrates that they allude as it were to the potential infinity of organic existence. The central part of the animal shown in this picture, a scolopendrid (Figure 28), possesses fairly regular translational, combined with bilateral, symmetry, the basic operations of which are translation by one segment and longitudinal reflection.

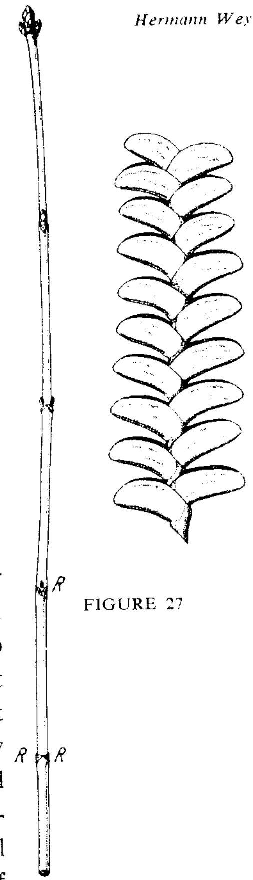

FIGURE 27

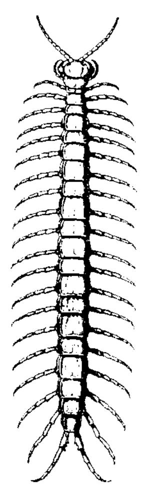

FIGURE 28

In one-dimensional *time* repetition at equal intervals is the musical principle of *rhythm*. As a shoot grows it translates, one might say, a slow temporal into a spatial rhythm. Reflection, inversion in time, plays a far less important part in music than rhythm does. A melody changes its character to a considerable degree if played backward, and I, who am a poor musician, find it hard to recognize reflection when it is used in the construction of a fugue; it certainly has no such spon-

[14] This and the next picture are taken from *Studium Generale*, p. 249 and p. 241 (article by W. Troll, "Symmetriebetrachtung in der Biologie").

taneous effect as rhythm. All musicians agree that underlying the emotional element of music is a strong formal element. It may be that it is capable of some such mathematical treatment as has proved successful for the art of ornaments. If so, we have probably not yet discovered the appropriate mathematical tools. This would not be so surprising. For after all, the Egyptians excelled in the ornamental art four thousand years before the mathematicians discovered in the group concept the proper mathematical instrument for the treatment of ornaments and for the derivation of their possible symmetry classes. Andreas Speiser, who has taken a special interest in the group-theoretic aspect of ornaments, tried to apply combinatorial principles of a mathematical nature also to the formal problems of music. There is a chapter with this title in his book, "Die mathematische Denkweise," (Zurich, 1932). As an example, he analyzes Beethoven's pastoral sonata for piano, opus 28, and he also points to Alfred Lorenz's investigations on the formal structure of Richard Wagner's chief works. Metrics in poetry is closely related, and here, so Speiser maintains, science has penetrated much deeper. A common principle in music and prosody seems to be the configuration *a a b* which is often called a bar: a theme *a* that is repeated and then followed by the "envoy" *b*; strophe, antistrophe, and epode in Greek choric lyrics. But such schemes fall hardly under the heading of symmetry.[15]

We return to symmetry in space. Take a band ornament where the individual section repeated again and again is of length a and sling it around a circular cylinder, the circumference of which is an integral multiple of a, for instance $25a$. You then obtain a pattern which is carried over into itself through the rotation around the cylinder axis by $\alpha = 360°/25$ and its repetitions. The twenty-fifth iteration is the rotation by 360°, or the identity. We thus get a finite group of rotations of order 25, i.e. one consisting of 25 operations. The cylinder may be replaced by any surface of cylindrical symmetry, namely by one that is carried into itself by all rotations around a certain axis, for instance by a vase. Figure 29 shows an attic vase of the geometric period which displays quite a number of simple ornaments of this type. The principle of symmetry is the same, although the style is no longer "geometric," in this Rhodian pitcher (Figure 30), Ionian school of the seventh century B.C. Other illustrations are such capitals as these from early Egypt (Figure 31). Any finite group of proper rotations around a point O in a plane, or around a given axis in space, contains a primitive rotation t whose angle is an aliquot part $360°/n$ of the full rotation by 360°, and consists of its iterations t^1, t^2, $\ldots, t^{n-1}$, t^n = identity. The order n completely characterizes this group. The result follows from the analogous fact that any group of translations

[15] The reader should compare what G. D. Birkhoff has to say on the mathematics of poetry and music in the two publications quoted in note 1.

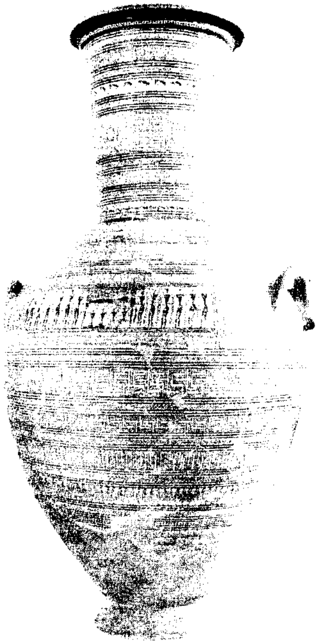

FIGURE 29

FIGURE 30

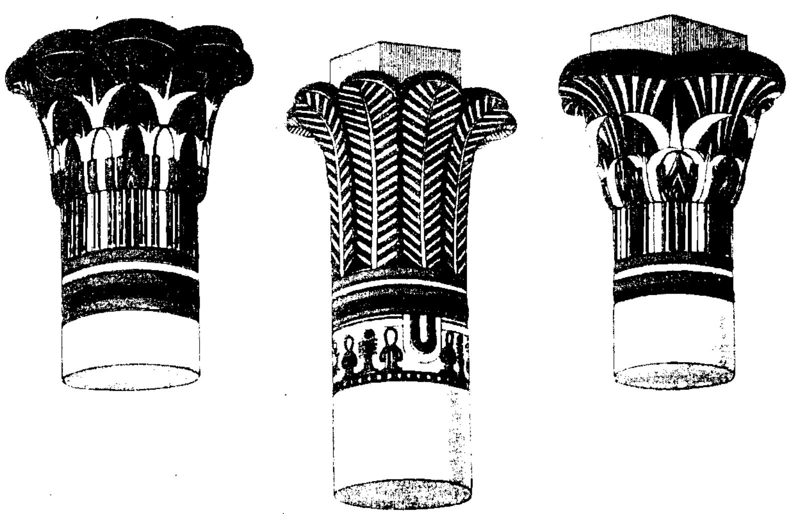

FIGURE 31

of a line, provided it contains no operations arbitrarily near to the identity except the identity itself, consists of the iterations va of a single translation $a(v = 0, \pm 1, \pm 2, \cdots)$.

The wooden dome in the Bardo of Tunis, once the palace of the Beys of Tunis (Figure 32), may serve as an example from interior architecture.

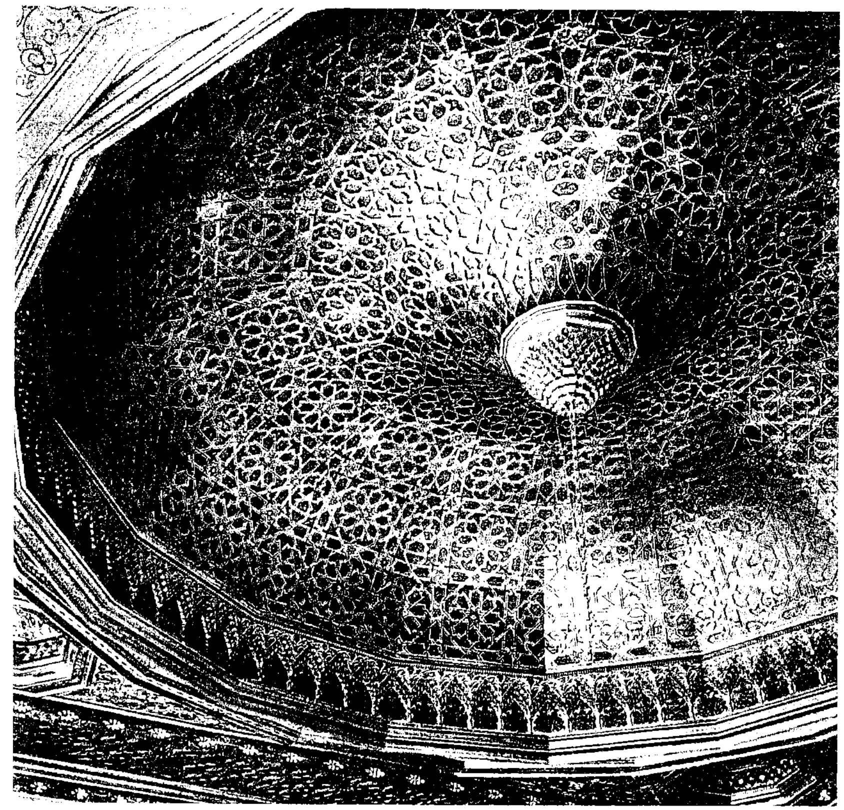

FIGURE 32

The next picture (Figure 33) takes you to Pisa: the Baptisterium with the tiny-looking statue of John the Baptist on top is a central building in whose exterior you can distinguish six horizontal layers each of rotary symmetry of a different order n. One could make the picture still more impressive by adding the leaning tower with its six galleries of arcades all having rotary symmetry of the same high order and the dome itself, the exterior of whose nave displays in columns and friezes patterns of the lineal translatory type of symmetry while the cupola is surrounded by a colonnade of high order rotary symmetry.

An entirely different spirit speaks to us from the view, seen from the rear of the choir, of the Romanesque cathedral in Mainz, Germany

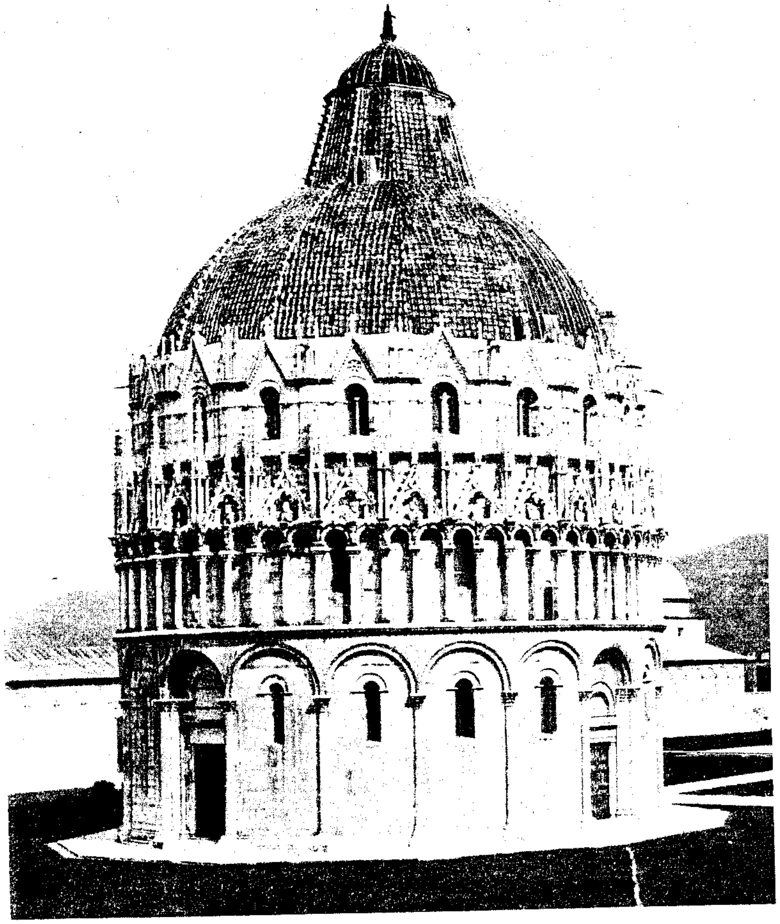

FIGURE 33

(Figure 34). Yet again repetition in the round arcs of the friezes, octagonal central symmetry ($n = 8$, a low value compared to those embodied in the several layers of the Pisa Baptisterium) in the small rosette and the three towers, while bilateral symmetry rules the structure as a whole as well as almost every detail.

Cyclic symmetry appears in its simplest form if the surface of fully cylindrical symmetry is a plane perpendicular to the axis. We then can limit ourselves to the two-dimensional plane with a center O. Magnificent examples of such central plane symmetry are provided by the rose windows of Gothic cathedrals with their brilliant-colored glasswork. The richest I remember is the rosette of St. Pierre in Troyes, France, which is based on the number 3 throughout.

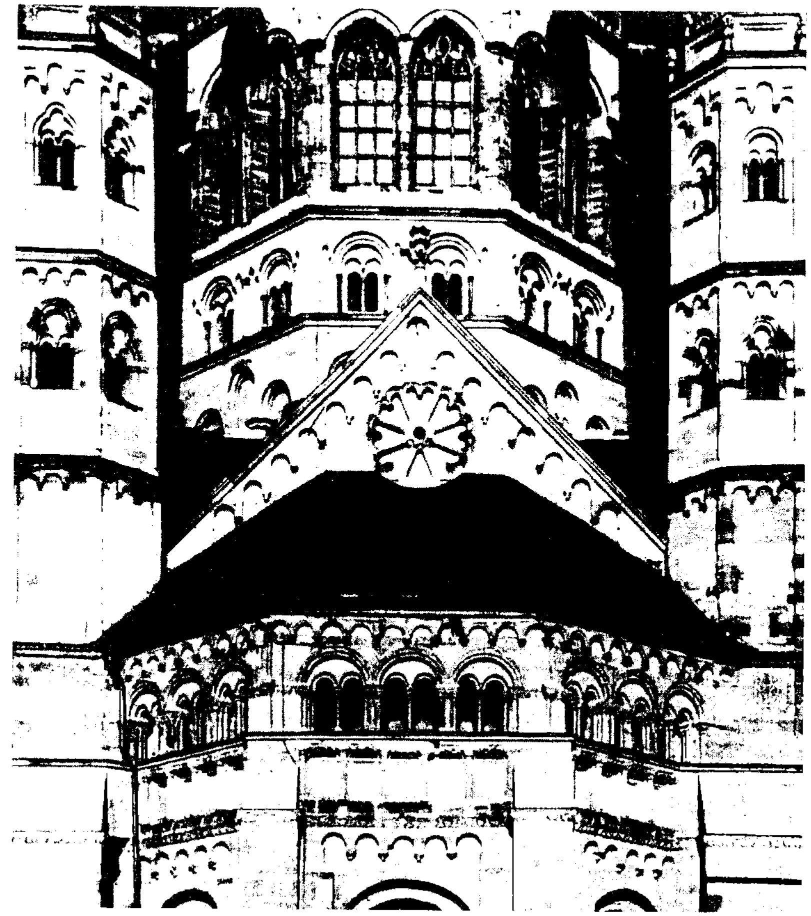

FIGURE 34

Flowers, nature's gentlest children, are also conspicuous for their colors and their cyclic symmetry. Here (Figure 35) is a picture of an iris with its triple pole. The symmetry of 5 is most frequent among flowers. A page like the following (Figure 36) from Ernst Haeckel's *Kunstformen der Natur* seems to indicate that it also occurs not infrequently among the lower animals. But the biologists warn me that the outward appearance of these echinoderms of the class of *Ophiodea* is to a certain degree deceptive; their larvae are organized according to the principle of bilateral symmetry. No such objection attaches to the next picture from the same source (Figure 37), a *Discomedusa* of octagonal symmetry. For the coelentera occupy a place in the phylogenetic evolution where cyclic has not yet given way to bilateral symmetry. Haeckel's extraordinary work, in which his interest in the concrete forms of organisms finds expression in

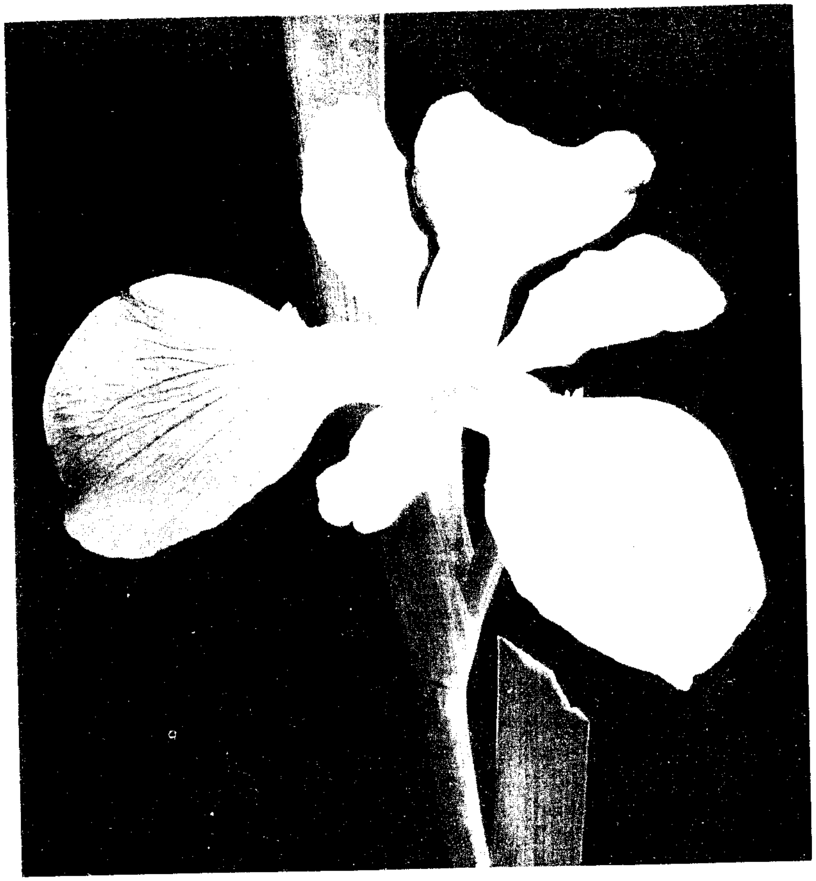

FIGURE 35

countless drawings executed in minutest detail, is a true nature's codex of symmetry. Equally revealing for Haeckel, the biologist, are the thousands and thousands of figures in his *Challenger Monograph*, in which he describes for the first time 3,508 new species of radiolarians discovered by him on the Challenger Expedition, 1887. One should not forget these accomplishments over the often all-too-speculative phylogenetic constructions in which this enthusiastic apostle of Darwinism indulged, and over his rather shallow materialistic philosophy of monism, which made quite a splash in Germany around the turn of the century.

Speaking of *Medusae* I cannot resist the temptation of quoting a few lines from D'Arcy Thompson's classic work on *Growth and Form*, a masterpiece of English literature, which combines profound knowledge in geometry, physics, and biology with humanistic erudition and scientific insight of unusual originality. Thompson reports on physical experiments

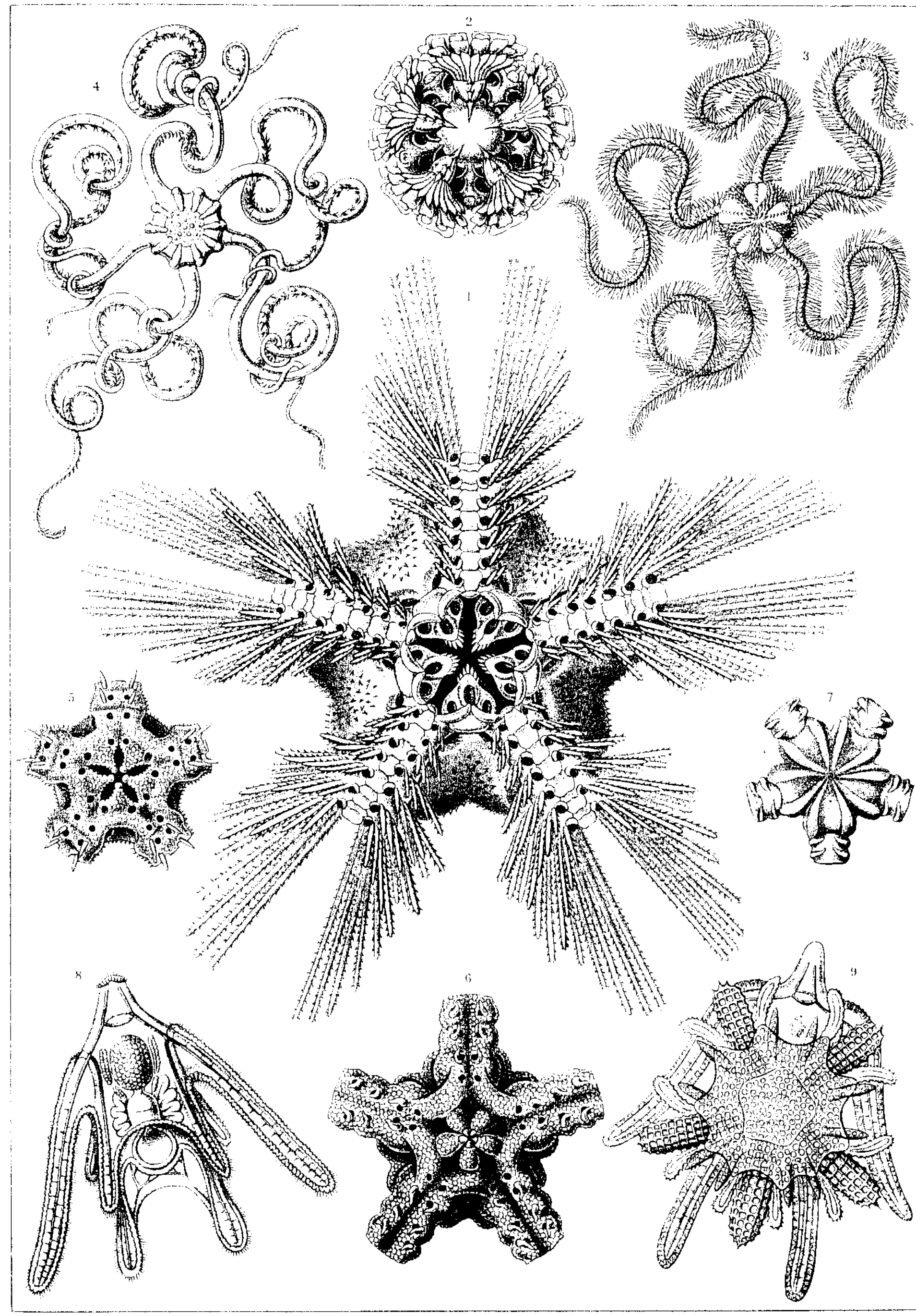

FIGURE 36

with hanging drops which serve to illustrate by analogy the formation of medusae. "The living medusa," he says, "has geometrical symmetry so marked and regular as to suggest a physical or mechanical element in the little creatures' growth and construction. It has, to begin with, its vortex-like bell or umbrella, with its symmetrical handle or manubrium. The bell is traversed by radial canals, four or in multiples of four; its edge is beset with tentacles, smooth or often beaded, at regular intervals or of graded sizes; and certain sensory structures, including solid concretions or 'otoliths,' are also symmetrically interspersed. No sooner made, then it begins to pulsate; the bell begins to 'ring.' Buds, miniature replicas of the parent-organism, are very apt to appear on the tentacles, or on the manubrium or sometimes on the edge of the bell; we seem to see one vortex producing others before our eyes. The development of a medusoid deserves to be studied without prejudice from this point of view. Certain it is that the tiny medusoids of *Obelia*, for instance, are budded off with a rapidity and a complete perfection which suggests an automatic and all but instantaneous act of conformation, rather than a gradual process of growth."

While pentagonal symmetry is frequent in the organic world, one does not find it among the most perfectly symmetrical creations of inorganic nature, among the crystals. There no other rotational symmetries are possible than those of order 2, 3, 4, and 6. Snow crystals provide the best known specimens of hexagonal symmetry. Figure 38 shows some of these little marvels of frozen water. In my youth, when they came down from heaven around Christmastime blanketing the landscape, they were the delight of old and young. Now only the skiers like them, while they have become the abomination of motorists. Those versed in English literature will remember Sir Thomas Browne's quaint account in his *Garden of Cyrus* (1658) of hexagonal and "quincuncial" symmetry which "doth neatly declare how nature Geometrizeth and observeth order in all things." One versed in German literature will remember how Thomas Mann in his *Magic Mountain* [16] describes the "hexagonale Unwesen" of the snow storm in which his hero, Hans Castorp, nearly perishes when he falls asleep with exhaustion and leaning against a barn dreams his deep dream of death and love. An hour before when Hans sets out on his unwarranted expedition on skis he enjoys the play of the flakes "and among these myriads of enchanting little stars," so he philosophizes, "in their hidden splendor, too small for man's naked eye to see, there was not one like unto another; an endless inventiveness governed the development and unthinkable differentiation of one and the same basic scheme, the equilateral, equiangled hexagon. Yet each in itself—this was the uncanny, the antiorganic, the

[16] I quote Helen Lowe-Porter's translation, Knopf, New York, 1927 and 1939.

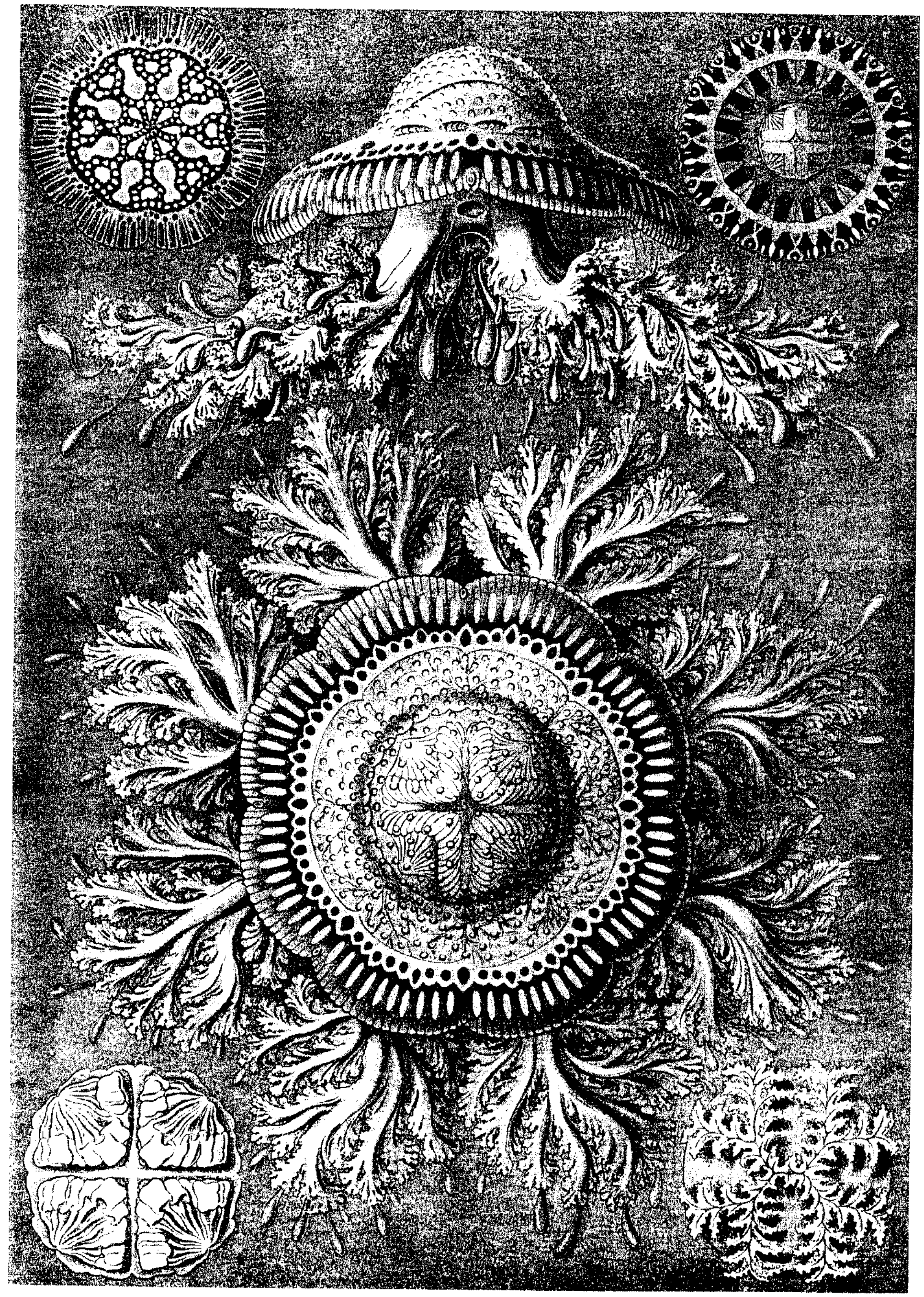

FIGURE 37

life-denying character of them all—each of them was absolutely symmetrical, icily regular in form. They were too regular, as substance adapted to life never was to this degree—the living principle shuddered at this perfect precision, found it deathly, the very marrow of death—Hans Castorp felt he understood now the reason why the builders of antiquity purposely and secretly introduced minute variation from absolute symmetry in their columnar structures."[17]

Up to now we have paid attention to proper rotations only. If improper rotations are taken into consideration, we have the two following possibilities for finite groups of rotations around a center O in plane geometry, which correspond to the two possibilities we encountered for ornamental symmetry on a line: (1) the group consisting of the repetitions of a single proper rotation by an aliquot part $\alpha = 360°/n$ of $360°$; (2) the group of these rotations combined with the reflections in n axes forming angles of $\frac{1}{2}\alpha$. The first group is called the cyclic group C_n and the second the dihedral group D_n. Thus these are the only possible central symmetries in two-dimensions:

(1) $$C_1, C_2, C_3, \cdots; D_1, D_2, D_3, \cdots.$$

C_1 means no symmetry at all, D_1 bilateral symmetry and nothing else. In architecture the symmetry of 4 prevails. Towers often have hexagonal symmetry. Central buildings with the symmetry of 6 are much less frequent. The first pure central building after antiquity, S. Maria degli Angeli in Florence (begun 1434), is an octagon. Pentagons are very rare. When once before I lectured on symmetry in Vienna in 1937 I said I knew of only one example and that a very inconspicuous one, forming the passageway from San Michele di Murano in Venice to the hexagonal Capella Emiliana. Now, of course, we have the Pentagon building in Washington. By its size and distinctive shape, it provides an attractive landmark for bombers. Leonardo da Vinci engaged in systematically determining the possible symmetries of a central building and how to attach chapels and niches without destroying the symmetry of the nucleus. In abstract modern terminology, his result is essentially our above table of the possible finite groups of rotations (proper and improper) in two dimensions.

So far the rotational symmetry in a plane had always been accompanied by reflective symmetry; I have shown you quite a number of examples for the dihedral group D_n and none for the simpler cyclic group C_n. But this is more or less accidental. Here (Figure 39) are two flowers, a *gera-*

[17] Dürer considered his canon of the human figure more as a standard from which to deviate than as a standard toward which to strive. Vitruvius' *temperaturae* seem to have the same sense, and maybe the little word "almost" in the statement ascribed to Polykleitos and mentioned in note 1 points in the same direction.

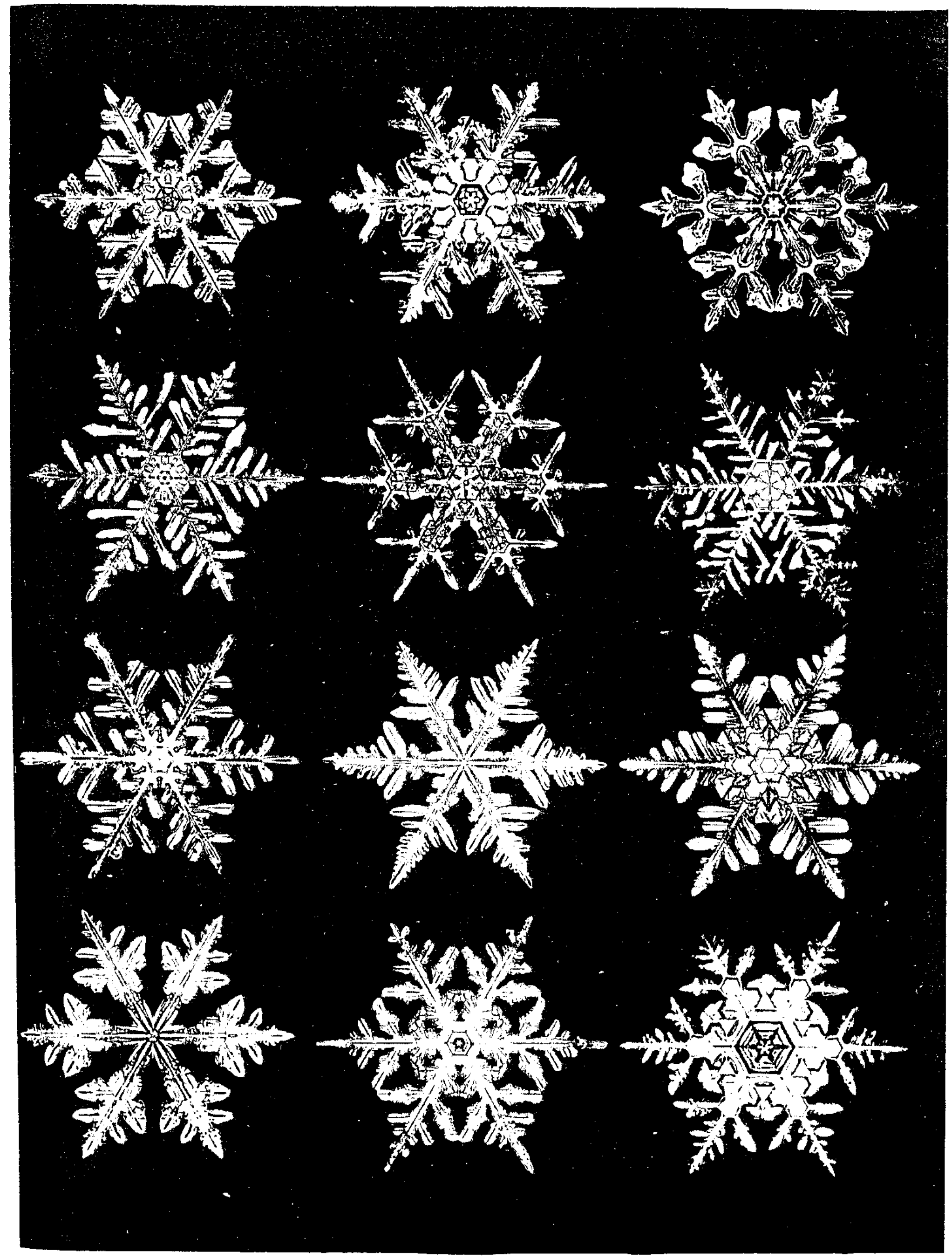

FIGURE 38

nium (I) with the symmetry group D_5 while *Vinca herbacea* (II) has the more restricted group C_5 owing to the asymmetry of its petals. Figure 40 shows what is perhaps the simplest figure with rotational symmetry, the tripod $(n = 3)$. When one wants to eliminate the attending reflective symmetry, one puts little flags unto the arms and obtains the triquetrum,

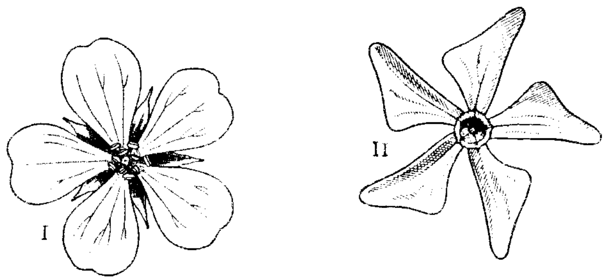

FIGURE 39

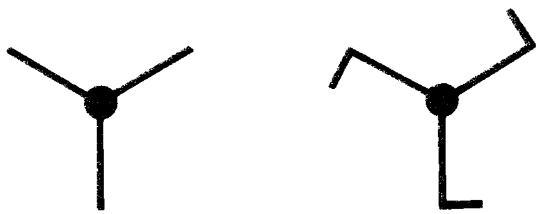

FIGURE 40

an old magic symbol. The Greeks, for instance, used it with the Medusa's head in the center as the symbol for the three-cornered Sicily. (Mathematicians are familiar with it as the seal on the cover of the *Rendiconti del Circolo Matematico di Palermo.*) The modification with four instead of three arms is the swastika, which need not be shown here—one of the most primeval symbols of mankind, common possession of a number of apparently independent civilizations. In my lecture on symmetry in Vienna in the fall of 1937, a short time before Hitler's hordes occupied Austria, I added concerning the swastika: "In our days it has become the symbol of a terror far more terrible than the snake-girdled Medusa's head"—and a pandemonium of applause and booing broke loose in the audience. It seems that the origin of the magic power ascribed to these patterns lies in their startling incomplete symmetry—rotations without reflections. Here (Figure 41) is the gracefully designed staircase of the pulpit of the Stephan's dome in Vienna; a triquetrum alternates with a swastika-like wheel.

So much about rotational symmetry in two dimensions. If dealing with

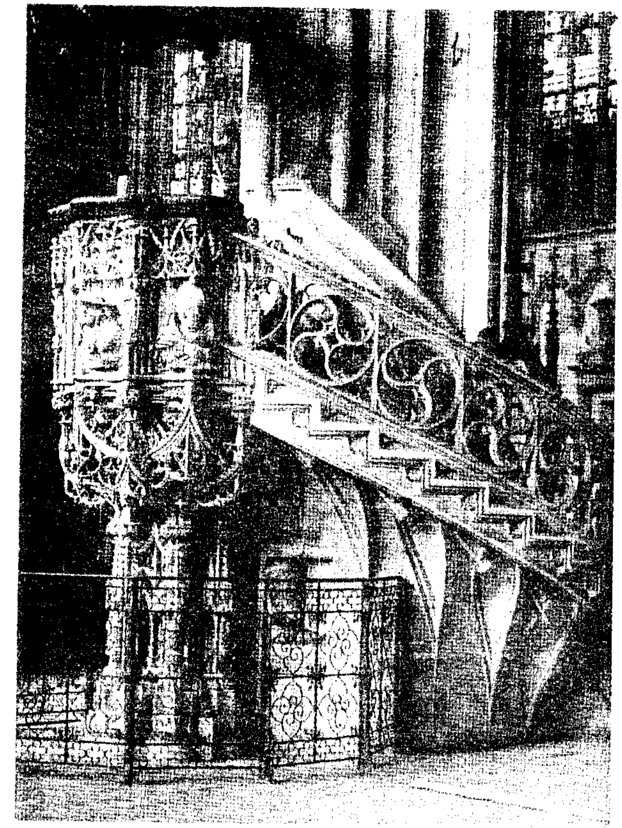

FIGURE 41

potentially infinite patterns like band ornaments or with infinite groups, the operation under which the pattern is invariant is not of necessity a congruence but could be a similarity. A similarity in one dimension that is not a mere translation has a fixed point O and is a dilatation s from O in a certain ratio $a : 1$ where $a \neq 1$. It is no essential restriction to assume $a > 0$. Indefinite iteration of this operation generates a group Σ consisting of the dilatations

$$s^n \quad (n = 0, \pm 1, \pm 2, \cdots). \tag{2}$$

A good example of this type of symmetry is shown by the shell of *Turritella duplicata* (Figure 42). It is really quite remarkable how exactly the widths of the consecutive whorls of this shell follow the law of geometric progression.

The hands of some clocks perform a continuous uniform rotation, others jump from minute to minute. The rotations by an integral number of

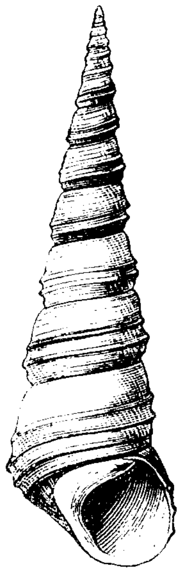

FIGURE 42

minutes form a discontinuous subgroup within the continuous group of all rotations, and it is natural to consider a rotation s and its iterations (2) as contained in the continuous group. We can apply this viewpoint to any similarity in 1, 2, or 3 dimensions, as a matter of fact to any transformation s. The continuous motion of a space-filling substance, a "fluid," can mathematically be described by giving the transformation $U(t,t')$ which carries the position P_t of any point of the fluid at the moment t over into its position $P_{t'}$ at the time t'. These transformations form a one-parameter group if $U(t,t')$ depends on the time difference $t'-t$ only, $U(t,t') = S(t'-t)$, i.e. if during equal time intervals always the same motion is repeated. Then the fluid is in "uniform motion." The simple group law

$$S(t_1)S(t_2) = S(t_1 + t_2)$$

expresses that the motions during two consecutive time intervals t_1, t_2 result in the motion during the time $t_1 + t_2$. The motion during 1 minute leads to a definite transformation $s = S(1)$, and for all integers n the motion $S(n)$ performed during n minutes is the iteration s^n: the discontinuous group Σ consisting of the iterations of s is embedded in the continuous group with the parameter t consisting of the motions $S(t)$. One could say that the continuous motion consists of the endless repetition of the same infinitesimal motion in consecutive infinitely small time intervals of equal length.

We could have applied this consideration to the rotations of a plane disc as well as to dilatations. We now envisage any proper similarity s, i.e. one which does not interchange left and right. If, as we assume, it is not a mere translation, it has a fixed point O and consists of a rotation about O combined with a dilatation from the center O. It can be obtained as the stage $S(1)$ reached after 1 minute by a continuous process $S(t)$ of combined uniform rotation and expansion. This process carries a point $\neq O$ along a so-called logarithmic or equiangular spiral. This curve, therefore, shares with straight line and circle the important property of going over into itself by a continuous group of similarities. The words by which James Bernoulli had the *spira mirabilis* adorned on his tombstone in the

Münster at Basle, "Eadem mutata resurgo," are a grandiloquent expression of this property. Straight line and circle are limiting cases of the logarithmic spiral, which arise when in the combination rotation-plus-dilatation one of the two components happens to be the identity. The stages reached by the process at the times

$$(3) \qquad t = n = \cdots, -2, -1, 0, 1, 2, \cdots$$

form the group consisting of the iterations (2). The well-known shell of *Nautilus* (Figure 43) shows this sort of symmetry to an astonishing per-

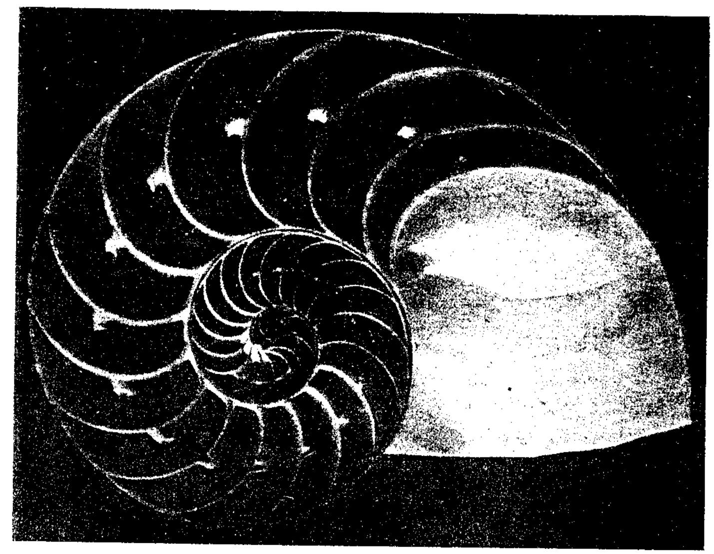

FIGURE 43

fection. You see here not only the continuous logarithmic spiral, but the potentially infinite sequence of chambers has a symmetry described by the discontinuous group Σ. For everybody looking at this picture (Figure 44) of a giant sunflower, *Helianthus maximus*, the florets will naturally arrange themselves into logarithmic spirals, two sets of spirals of opposite sense of coiling.

The most general rigid motion in three-dimensional space is a screw motion s, combination of a rotation around an axis with a translation along that axis. Under the influence of the corresponding continuous uniform motion any point not on the axis describes a screw-line or helix which, of course, could say of itself with the same right as the logarithmic spiral: *eadem resurgo.* The stages P_n which the moving point reaches at the equidistant moments (3) are equidistributed over the helix like stairs on a winding staircase. If the angle of rotation of the operation s is a fraction μ/ν of the full angle 360° expressible in terms of small integers

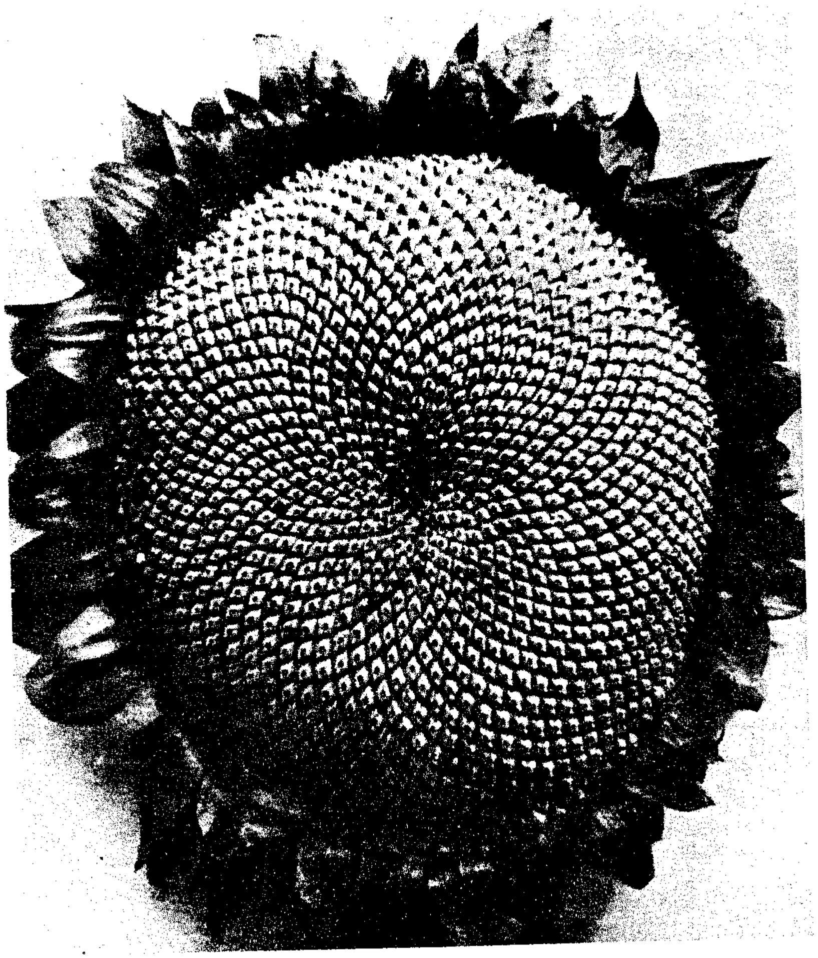

FIGURE 44

μ, ν then every νth point of the sequence P_n lies on the same vertical, and μ full turnings of the screw are necessary to get from P_n to the point $P_{n+\nu}$ above it. The leaves around the shoot of a plant often show such a regular spiral arrangement. Goethe spoke of a spiral tendency in nature, and under the name of *phyllotaxis* this phenomenon, since the days of Charles Bonnet (1754), has been the subject of much investigation and more speculation among botanists.[18] One has found that the fractions μ/ν representing the screw-like arrangement of leaves quite often are members of the "Fibonacci sequence"

[18] This phenomenon plays also a role in J. Hambidge's constructions. His *Dynamic symmetry* contains on pp. 146 157 detailed notes by the mathematician R. C. Archibald on the logarithmic spiral, golden section, and the Fibonacci series.

$$(4) \qquad 1/1,\ 1/2,\ 2/3,\ 3/5,\ 5/8,\ 8/13,\ 13/21,\ 21/34,\ \cdots,$$

which results from the expansion into a continued fraction of the irrational number $\frac{1}{2}(\sqrt{5}-1)$. This number is no other but the ratio known as the *aurea sectio*, which has played such a role in attempts to reduce beauty of proportion to a mathematical formula. The cylinder on which the screw is wound could be replaced by a cone; this amounts to replacing the screw motion s by any proper similarity—rotation combined with dilatation. The arrangement of scales on a fir-cone falls under this slightly more general form of symmetry in phyllotaxis. The transition from cylinder over cone to disc is obvious, illustrated by the cylindrical stem of a plant with its leaves, a fir-cone with its scales, and the discoidal inflorescence of *Helianthus* with its florets. Where one can check the numbers (4) best, namely for the arrangement of scales on a fir-cone, the accuracy is not too good nor are considerable deviations too rare. P. G. Tait, in the *Proceedings of the Royal Society of Edinburgh* (1872), has tried to give a simple explanation, while A. H. Church in his voluminous treatise *Relations of phyllotaxis to mechanical laws* (Oxford, 1901–1903) sees in the arithmetics of phyllotaxis an organic mystery. I am afraid modern botanists take this whole doctrine of phyllotaxis less seriously than their forefathers.

Apart from reflection all symmetries so far considered are described by a group consisting of the iterations of one operation s. In one case, and that is undoubtedly the most important, the resulting group is finite, namely if one takes for s a rotation by an angle $\alpha = 360°/n$ which is an aliquot part of the full rotation 360°. For the two-dimensional plane there are no other finite groups of proper rotations than these; witness the first line, $C_1, C_2, C_3, \cdots$ of Leonardo's table (1). The simplest figures which have the corresponding symmetry are the regular polygons: the regular triangle, the square, the regular pentagon, etc. The fact that there is for every number $n = 3, 4, 5, \cdots$ a regular polygon of n sides is closely related to the existence for every n of a rotational group of order n in plane geometry. Both facts are far from trivial. Indeed, the situation in three dimensions is altogether different: there do not exist infinitely many regular polyhedra in 3-space, but not more than five, often called the Platonic solids because they play an eminent role in Plato's natural philosophy. They are the regular tetrahedron, the cube, the octahedron, moreover the pentagondodecahedron, the sides of which are twelve regular pentagons, and the icosahedron bounded by twenty regular triangles. One might say that the existence of the first three is a fairly trivial geometric fact. But the discovery of the last two is certainly one of the most beautiful and singular discoveries made in the whole history of mathematics. With a fair amount of certainty, it can be traced to the colonial Greeks in southern Italy. The suggestion has been made that they abstracted the regular dodecahedron from the crystals of pyrite, a sulphurous mineral abundant in Sicily. But as mentioned before, the symmetry of 5 so characteristic for the regular dodecahedron contradicts the laws of crystallography, and indeed one finds that the pentagons bounding the dodecahedra in which pyrite crystallizes have 4 edges of equal, but one of different, length. The first exact construction of the regular pentagondodecahedron is probably due to Theaetetus. There is some evidence that dodecahedra were used as dice in Italy at a very early time and had some religious significance in Etruscan culture. Plato, in the dialogue *Timaeus*, associates the regular pyramid, octahedron, cube, icosahedron, with the four elements of fire, air, earth, and water (in this order), while in the pentagondodecahedron he sees in some sense the image of the universe as a whole. A. Speiser has advocated the view that the construction of the five regular solids is the chief goal of the deductive system of geometry as erected by the Greeks and canonized in Euclid's *Elements*. May I mention, however, that the Greeks never used the word "symmetric" in our modern sense. In common usage σύμμετρος means *proportionate*, while in Euclid it is equivalent to our *commensurable*: side and diagonal of a square are incommensurable quantities, ἀσύμμετρα μεγέθη.

Here (Figure 45) is a page from Haeckel's *Challenger Monograph* showing the skeletons of several Radiolarians. Numbers 2, 3, and 5 are octahedron, icosahedron, and dodecahedron in astonishingly regular form; 4 seems to have a lower symmetry.

Kepler, in his *Mysterium cosmographicum*, published in 1595, long before he discovered the three laws bearing his name today, made an attempt to reduce the distances in the planetary system to regular bodies which are alternatingly inscribed and circumscribed to spheres. Here (Figure 46) is his construction, by which he believed he had penetrated deeply into the secrets of the Creator. The six spheres correspond to the six planets, Saturn, Jupiter, Mars, Earth, Venus, Mercurius, separated in this order by cube, tetrahedron, dodecahedron, octahedron, icosahedron. (Of course, Kepler did not know about the three outer planets, Uranus, Neptune, and Pluto which were discovered in 1781, 1846, and 1930 respectively.) He tries to find the reasons why the Creater had chosen this order of the Platonic solids and draws parallels between the properties of the planets (astrological rather than astrophysical properties) and those of the corresponding regular bodies. A mighty hymn in which he proclaims his credo, "Credo spatioso numen in orbe," concludes his book. We still share his belief in a mathematical harmony of the universe. It has withstood the test of ever widening experience. But we no longer seek this harmony in static forms like the regular solids, but in dynamic laws.

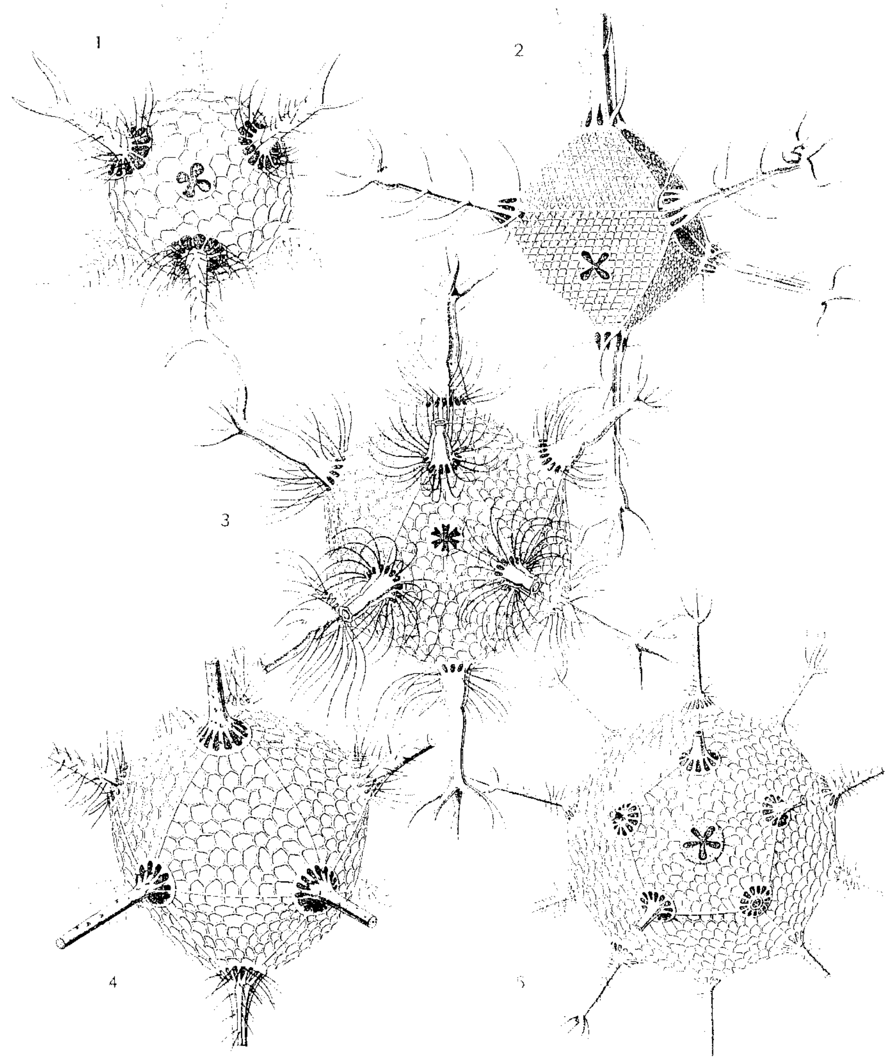

FIGURE 45

As the regular polygons are connected with the finite groups of plane rotations, so must the regular polyhedra be intimately related to the finite groups of proper rotations around a center O in space. From the study of plane rotations we at once obtain two types of proper rotation groups in space. Indeed, the group C_n of proper rotations in a horizontal plane around a center O can be interpreted as consisting of rotations in space around the vertical axis through O. Reflection of the horizontal plane in a line l of the plane can be brought about in space through a rotation

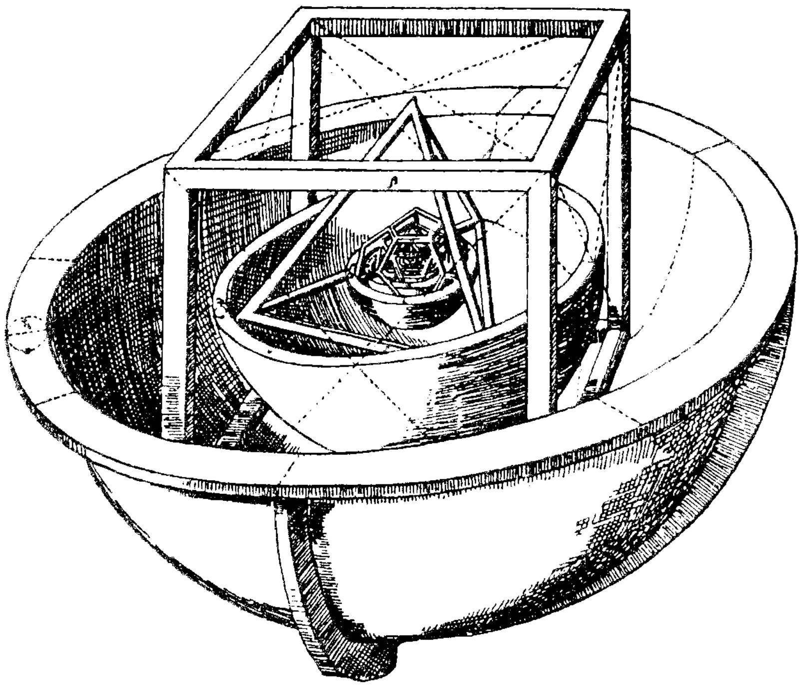

FIGURE 46

around l by 180° (*Umklappung*). You may remember that we mentioned this in connection with the analysis of a Sumerian picture (Figure 4). In this way the group D_n in the horizontal plane is changed into a group D'_n of proper rotations in space; it contains the rotations around a vertical axis through O by the multiples of $360^\circ/n$ and the Umklappungen around n horizontal axes through O which form equal angles of $360^\circ/2n$ with each other. But it should be observed that the group D'_1 as well as C_2 consists of the identity and the Umklappung around one line. These two groups are therefore identical, and in a complete list of the *different* groups of proper rotations in three dimensions D'_1 should be omitted if C_2 is kept. Hence we start our list thus:

$$C_1,\ C_2,\ C_3,\ C_4,\ \cdots;$$
$$D'_2,\ D'_3,\ D'_4,\ \cdots.$$

D'_2 is the so-called four-group consisting of the identity of the Umklappungen around three mutually perpendicular axes.

For each one of the five regular bodies we can construct the group of those proper rotations which carry that body into itself. Does this give rise to five new groups? No, only to three, and that for the following

reason. Inscribe a sphere into a cube and an octahedron into the sphere such that the corners of the octahedron lie where the sides of the cube touch the sphere, namely in the centers of the six square sides. (Figure 47 shows the two-dimensional analogue.) In this position cube and octa-

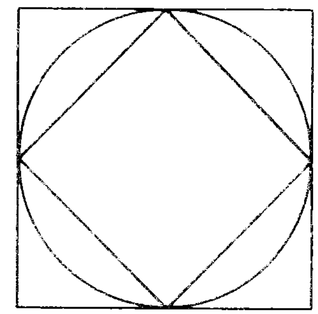

FIGURE 47

hedron are polar figures in the sense of projective geometry. It is clear that every rotation which carries the cube into itself also leaves the octahedron invariant, and vice versa. Hence the group for the octahedron is the same as for the cube. In the same manner pentagondodecahedron and icosahedron are polar figures. The figure polar to a regular tetrahedron is a regular tetrahedron the corners of which are the antipodes of those of the first. Thus we find three new groups of proper rotations, T, W, and P; they are those leaving invariant the regular tetrahedron, the cube (or octahedron), and the pentagondodecahedron (or icosahedron) respectively. Their orders, i.e. the number of operations in each of them, are 12, 24, 60 respectively.

It can be shown by a relatively simple analysis that with the addition of these three groups our table is complete:

$$\begin{array}{ll} C_n & (n = 1, 2, 3, \cdots), \\ D'_n & (n = 2, 3, \cdots); \\ T, W, P. & \end{array} \tag{5}$$

This is the modern equivalent to the tabulation of the regular polyhedra by the Greeks. These groups, in particular the last three, are an immensely attractive subject for geometric investigation.

What further possibilities arise if improper rotations are also admitted to our groups? This question is best answered by making use of one quite singular improper rotation, namely reflection in O: it carries any point P into its antipode P' with respect to O found by joining P with O and prolonging the straight line PO by its own length: $PO = OP'$. This operation Z commutes with every rotation S, $ZS = SZ$. Now let Γ be one of our finite groups of proper rotations. One way of including improper

rotations is simply by adjoining Z, more precisely by adding to the r
rotations S of Γ all the improper rotations of the form ZS (with S i
The order of the group $\Gamma = \Gamma + Z\Gamma$ thus obtained is clearly twice tl
Γ. Another way of including improper rotations arises from this situ
Suppose Γ is contained as a subgroup of index 2 in another group
proper rotations: so that one-half of the elements of Γ' lie in Γ, call
S, and one-half, S', do not. Now replace these latter by the improper
tions ZS'. In this manner you get a group $\Gamma'\Gamma$ which contains Γ whi
other half of its operations are improper. For instance, $\Gamma = C_n$ is a
group of index 2 of $\Gamma' = D'_n$; the operations S' of D'_n not contain
C_n are the Umklappungen around the n horizontal axes. The corres
ing ZS' are the reflections in the vertical planes perpendicular to
axes. Thus D'_nC_n consists of the rotations around the vertical axi
angles of which are multiples of $360°/n$, and of the reflections in ve
planes through this axis forming angles of $360°/2n$ with each other.
might say that this is the group formerly denoted by D_n. Another e
ple, the simplest of all: $\Gamma = C_1$ is contained in $\Gamma' = C_2$. The one oper
S' of C_2 not contained in C_1 is the rotation by $180°$ about the ve
axis; ZS' is reflection in the horizontal plane through O. Hence C_2
the group consisting of the identity and of the reflection in a given p
in other words, the group to which bilateral symmetry refers.

The two ways described are the only ones by which improper rota
may be included in our groups. Hence this is the complete table c
finite groups of (proper and improper) rotations:

$$
\begin{array}{llll}
C_n, & \overline{C}_n, & C_{2n}C_n & (n = 1, 2, 3, \cdots) \\
D'_n, & \overline{D}'_n, & D'_nC_n, \quad D'_{2n}D'_n & (n = 2, 3, \cdots) \\
T, \; W, \; P; & \overline{T}, \; \overline{W}, \; \overline{P}; & WT. &
\end{array}
$$

The last group WT is made possible by the fact that the tetrahedral g
T is a subgroup of index 2 of the octahedral group W.