

Deadline: 12N, Friday, 23 February

Late deadline: 12N, Friday, 9 March

Writing

- Using *clear sentences*, incorporate symbols, calculations, and, above all, illustrations into the text. Have an audience in mind. *Explain* what you're doing and why you're doing it—that is, provide *concise* and *coherent* arguments for what you're doing.
- *Ethic*: You may consult classmates and the instructor, but what you write should be *entirely your own work*.
- Typed work is appreciated. Hand-drawn figures are acceptable. Submit in hard copy.

1) Square and cube.

- a) Describe *all* of the *rotational* symmetries of a square. Consider only rotations that occur *in* the plane. (Recall that such a rotation is specified by a center and an amount of turning.)
- b) Label the vertices of the square a , b , c , and d . For each rotational symmetry, express the *permutation* of the vertices produced by the rotation.
- c) Describe *all* of the *rotational* symmetries of a cube. (Recall that a rotation in 3D is specified by a axis and an amount of turning.)
- d) Find *four* “things” that are permuted when the cube undergoes a rotational symmetry. (Of course, the things can't be the vertices, edges, or faces. But, they could involve combinations of these elements.)
- e) Label the four things a , b , c , and d . For each rotational symmetry of the cube, express the *permutation* of the four things produced by the rotation.

2) Composing isometries Set the following notation.

S_L is a reflection through the line L .

$R_{a,\alpha}$ is a rotation about a by an angle α .

$T_{\vec{v}}$ is a translation along the vector \vec{v} .

The *product* ST means: first apply T , then apply S .

Describe the following isometries in *simpler* terms. That is, find a more familiar transformation that does the same thing as the one that's given. For instance, two reflections in intersecting mirrors give a single rotation about the intersection point by an angle that's twice the angle between the mirrors.

- a) $S_L R_{a,\alpha}$ where L is the line $y = -x$, $a = (1, -1)$, and $\alpha = -30^\circ$
- b) $T_{\vec{v}} R_{a,\alpha}$ where $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $a = (1, 1)$, and $\alpha = 90^\circ$
- c) Can the composition of two translations ever produce something other than a translation? If so, give an example. If not, prove it.
- d) Can the composition of two rotations ever produce something other than a rotation? If so, give an example. If not, prove it. What transformations can a composition of two rotations produce?

Suggestion: Recall that a rotation or translation can be expressed as a product of two reflections. Can you simplify things by first making them more complicated? Can judicious choices for decomposing rotations and translations into products of reflections lead to something simpler?

- 3) **A glide reflection's mirror.** Consider a *general* glide reflection $G = S_M T_{\vec{v}}$ where S_M is reflection through the line M and $T_{\vec{v}}$ is translation along the vector \vec{v} . Find a line L that G sends to itself; that is, $G(L) = L$. (*Note:* L must be uniquely—*the fancy term is ‘canonically’*—determined by the data that specify the transformations that form the glide reflection.) What happens to the points on L ? How do they move? Describe the action of G restricted to L as an isometry of the line.

Bonus: Describe G as $S_L F$ where S_L is reflection through L and F is an isometry of the plane. That is, determine how F behaves. Notice that it now follows that L is the only line that is invariant under G .

- 4) **Composing reflections and rotations in 3 dimensions.** Let S denote a sphere in \mathbf{R}^3 .
- Determine the familiar transformation of 3-space that results when two reflections that preserve S are performed successively.
 - Determine the familiar transformation of 3-space that results when two rotations that preserve S are performed successively. (*Suggestion:* Decompose each rotation into a composition of reflections. The choice of mirrors is crucial.)
- 5) **Isometries of the sphere.** Following the approach we took in class for the plane, classify *all* of the isometries of the sphere—using angular separation about the sphere's center as a measure of length.

In-class presentations

- Elementary properties of isometries.** Show that an isometry of the plane is continuous, one-to-one, and onto. (You needn't make an ϵ - δ argument.)
- Composing isometries.** Describe the following isometries in *simpler* terms. That is, find a more familiar transformation that does the same thing as the one that's given.

a) $T_{\vec{v}} T_{\vec{u}}$ where $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

b) $T_{\vec{v}} S_L$ where $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and L is given by $y = -x + 1$.

- Commuting isometries.**

- a) State *necessary and sufficient* conditions on L and M so that

$$S_M S_L = S_L S_M.$$

- b) For $a \neq b$, state *necessary and sufficient* conditions on α and β so that

$$R_{a,\alpha} R_{b,\beta} = R_{b,\beta} R_{a,\alpha}.$$

- Isometries of the line and circle.**

- Classify isometries of the line. Treat the line *intrinsically*—not as an object sitting inside another space, such as the plane.
- Using angle measure as a way of measuring distance—a *metric*—on the circle, classify the isometries of the circle. Again, treat the circle in intrinsic terms.