Exploration 1 **Math 355** **Fall 2014**
Deadline: 12N, Friday, 26 September

Writing

- Using *clear sentences*, incorporate symbols, calculations, and, above all, illustrations into the text. Have an audience in mind. *Explain* what you're doing and why you're doing it—that is, provide *concise* and *coherent* arguments for what you're doing.
- *Ethic*: You may consult classmates and the instructor, but what you write should be *entirely your own work*.
- Typed work is appreciated. Hand-drawn figures are acceptable.

1) **Square and cube.**

   a) Describe *all* of the *rotational* symmetries of a square. Consider only rotations that occur *in* the plane. (Recall that such a rotation is specified by a center and an amount of turning.)
   b) Label the vertices of the square a, b, c, and d. For each rotational symmetry, express the *permutation* of the vertices produced by the rotation.
   c) Describe *all* of the *rotational* symmetries of a cube. (Recall that a rotation in 3D is specified by a axis and an amount of turning.)
   d) Find *four* "things" that are permuted when the cube undergoes a rotational symmetry. (Of course, the things can't be the vertices, edges, or faces. But, they could involve combinations of these elements.)
   e) Label the four things a, b, c, and d. For each rotational symmetry of the cube, express the *permutation* of the four things produced by the rotation.

2) **Composing isometries** Set the following notation.

   S_L is a reflection through the line L.
   $R_{a,\alpha}$ is a rotation about a by an angle α.
   $T_{\vec{v}}$ is a translation along the vector $\vec{v}$.
   The *product* ST means: first apply T, then apply S.

   Describe the following isometries in *simpler* terms. That is, find a more familiar transformation that does the same thing as the one that's given. For instance, two reflections in intersecting mirrors give a single rotation about the intersection point by an angle that's twice the angle between the mirrors.

   a) $S_L\, R_{a,\alpha}$ where L is the line $y = -x$, $a = (1, -1)$, and $\alpha = -30°$
   b) $T_{\vec{v}}\, R_{a,\alpha}$where $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $a = (1, 1)$, and $\alpha = 90°$
   c) Can the composition of two translations ever produce something other than a translation? If so, give an example. If not, prove it.
   d) Can the composition of two rotations ever produce something other than a rotation? If so, give an example. If not, prove it. What transformations can a composition of two rotations produce?

   Suggestion: Recall that a rotation or translation can be expressed as a product of two reflections. Can you simplify things by first making them more complicated? Can judicious choices for decomposing rotations and translations into products of reflections lead to something simpler?

3) **A glide reflection's mirror.** Consider a *general* glide reflection $G = S_M T_{\vec{v}}$ where S_M is reflection through the line M and $T_{\vec{v}}$ is translation along the vector $\vec{v}$. Find a line L that G sends to intself; that is, $G(L) = L$. (*Note:* L must be uniquely—the fancy term is 'canonically'—determined by the data that specify the transformations that form the glide reflection.) What happens to the points on L? How do they move? Describe the action of G restricted to L as an isometry of the the line.

   Bonus: Describe G as $S_L F$ where S_F is reflection through L and F is an isometry of the plane. Notice that it now follows that L is the only line that is invariant under G.

4) **Composing reflections and rotations in 3 dimensions.** Let S denote a sphere in $\mathbf{R}^3$.

   a) Determine the familiar transformation of 3-space that results when two reflections that preserve S are performed successively.
   b) Determine the familiar transformation of 3-space that results when two rotations that preserve S are performed successively. (*Suggestion:* Decompose each rotation into a composition of reflections. The choice of mirrors is crucial.)

5) **Isometries of the sphere.** Following the approach we took in class for the plane, classify *all* of the isometries of the sphere—using angular separation about the sphere's center as a measure of length.

In-class presentations

1) Show that an isometry of the plane is continuous, one-to-one, and onto. (You needn't make an ϵ-δ argument.)
2) **Composing isometries.** Describe the following isometries in *simpler* terms. That is, find a more familiar transformation that does the same thing as the one that's given.

   a) $T_{\vec{v}} T_{\vec{u}}$ where $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.
   b) $T_{\vec{v}} S_L$ where $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and L is given by $y = -x + 1$.

3) **Commuting isometries.**

   a) State *necessary and sufficient* conditions on L and M so that
   $$S_M S_L = S_L S_M.$$
   b) For $a \neq b$, state *necessary and sufficient* conditions on α and β so that
   $$R_{a,\alpha} R_{b,\beta} = R_{b,\beta} R_{a,\alpha}.$$

4) **Isometries of the line and circle.**

   a) Classify isometries of the line. Treat the line *intrinsically*—not as an object sitting inside another space, such as the plane.
   b) Using angle measure as a way of measuring distance—a *metric*—on the circle, classify the isometries of the circle. Again, treat the circle in intrinsic terms.