Exploration 2 **Math 355** **Spring 2019**
Deadline: 12N Friday 12 April
Late deadline: 12N, Friday, 26 April

Writing

- Using *concise and clear sentenceLs*, incorporate symbols, calculations, and, above all, illustrations into the text. Have an audience in mind. *Explain* what you're doing and why you're doing it.
- You may consult classmates and the instructor, but what you write should be *entirely your own work.*
- Typed work is appreciated. Hand-drawn figures are acceptable.

1) **Classifying polyhedra.** Two polygons or two polyhedra have the same *type* if one can be turned into the other by continuously deforming the edges (stretching or shrinking) without collapsing an edge or eliminating a vertex.

   To classify polygons by type, we can use vertices and edges. There's only one type of polygon with n vertices and n edges. (Indeed, just one of these numbers is enough.)

   Can we use vertices, edges, and faces to classify spherical polyhedra by type? That is, given $V,\ E,\ F$ such that $V - E + F = 2$, is there *at most* one type of polyhedron with V vertices, E edges, and F faces? Looking for simple examples to show that this doesn't happen might be a good first step.

   Now, suppose we have three number $x, y, z \geq 4$ such that $x - y + z = 2$. Is there a polyhedron for which $V = x,\ E = y,\ F = z$?

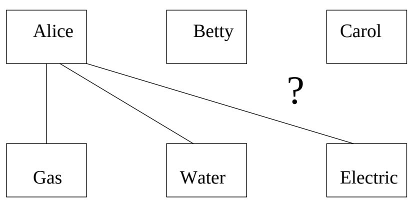
2) **Polyhedra with seven or eight edges.** Can a spherical polyhedron have seven edges? If so, find all of the types. If not, prove it. Same question for eight edges.

3) **Five-faced polyhedra.** Two polyhedra P and Q are of the same *type* if they describe equivalent networks of vertices and edges. That is, you can turn P into Q by stretching and shrinking (but not cutting) edges.

   a) Use the euler number, polyhedron inequalities, and $V,\ E,\ F$ numerology to describe *every type* of sphere-like polyhedra with five faces.

      Note: Be careful not only to construct the constructible polyhedra, but to argue that they are the only constructible forms.

   b) Use this result to show that you can't connect three houses to three utilities (gas, water, and electric) *without* having the connections cross.



   c) Suppose Alice, Betty, and Carol live on a torus (donut surface). Show that their houses *can* be connected to the three utilities without the connections crossing. Where does your argument above break down?

4) **What can't make a spherical polyhedron** Use the Euler number to prove that there's no spherical polyhedron composed of triangular faces meeting six at *each* vertex. Note: it follows that no regular spherical polyhedron can have seven triangles at a vertex.

   Generalize the argument to the following cases.

   a) k triangles per vertex where $k > 6$
   b) k quadrilaterals per vertex where $k > 4$
   c) k pentagons per vertex where $k > 3$
   d) k n-gons per vertex where $k > 3,\ n > 5$.

5) **Exploring frieze patterns**

   a) Design two frieze patterns that have the *same* symmetries, but look different in terms of the design.
   b) In addition to its fundamental translational symmetry, a frieze pattern can have one or more of four symmetries: rotational, reflection through a mirror *along* the strip, reflection through a mirror *across* the strip, and glide reflection. List the sixteen possible combinations of frieze symmetries. Which ones aren't actually possible to obtain? Which ones remain?

6) **Essence of a cube** Find a fundamental domain for the cube under its rotational symmetries and describe the quotient orbifold. (Use the same conditions as in the wallpaper case to assign an orbifold code.) Find a fundamental domain and quotient orbifold with reflective symmetries included. Do the same for the regular octahedron. Discuss how the two cases compare.

In-class talks

1) **Look-alike polyhedra.** Recall what it means for the vertices of a polyhedron to "look the same." Find a polyhedron in which all faces are congruent to one another and all vertices look the same, but the faces are not regular. Now find a polyhedron where all faces are regular (but, not the same shape) and every vertex looks the same. Try to find the *simplest* examples.

2) **Six-faced polyhedra.** Use the euler number, the polyhedron inequality, and V, E, F numerology to describe *all* of the types of sphere-like polyhedra with six faces.

   Suggestion: Consider how you can make a six-faced polyhedron out of a five-faced one.

3) **Polyhedral baseball.** Describe the symmetries of a baseball (ignore the stitches—consider only the seam formed by the two pieces of the cover. Now, build a polyhedron that has *exactly* the same symmetries as a baseball.

4) **Polyhedron inequalities.** In class we used the Euler number and relationships between faces, vertices and, edges to derive an inequality that describes a condition that the faces of a sphere-like polyhedron must satisfy:

$$3\,F_3 + 2\,F_4 + F_5 \geq 12.$$

   a) Is there a similar inequality that the vertices must satisfy? If so, what is it? If not, why not?
   b) Develop a polyhedron inequality—either for faces or vertices—for a torus-like polyhedron. How informative is it?