Exploration 3 Math 355 Spring 2019

Due: 12N Friday 10 May

Writing

- Select one item to write up. The score will be added to your overall exploration mark as a bonus.
- Using *concise and clear sentences*, incorporate symbols, calculations, and, above all, illustrations into the text. Have an audience in mind. *Explain* what you're doing and why you're doing it.
- You may consult classmates and the instructor, but what you write should be *entirely your own work.*
- Typed work is appreciated. Hand-drawn figures are acceptable.

1) Given a triangle ABC, call a hub point H a *trisection point* if the three angles formed by AH, BH, and CH are equal.

   For a triangle with no angle greater than $120°$, show that the network with a trisecting hub gives a *strictly* shorter total length than the shortest path *along* the edges.

   Hint: Extend the spokes AH, BH, CH to lines. Show that these lines bisect the $120°$ hub angles. From any vertex—say, A, drop perpendicular segments to the lines through BH and CH. Use the properties of this special configuration to compare the relevant lengths.

2) Suppose you want to enclose a *fixed area* with some fence? How do you do it in such a way that the length of fence is minimized? Is there a connection between the two cases: fixed perimeter-maximize area and fixed area-minimize perimeter? Can you solve the one problem if you know the solution to the other? If so, show how one solution implies the other.

In-class talks

1) What's the shortest path connecting two points on a sphere? A cylinder? Specify *how* you can find the path as well as how you know that it's the shortest. Compare the cases.

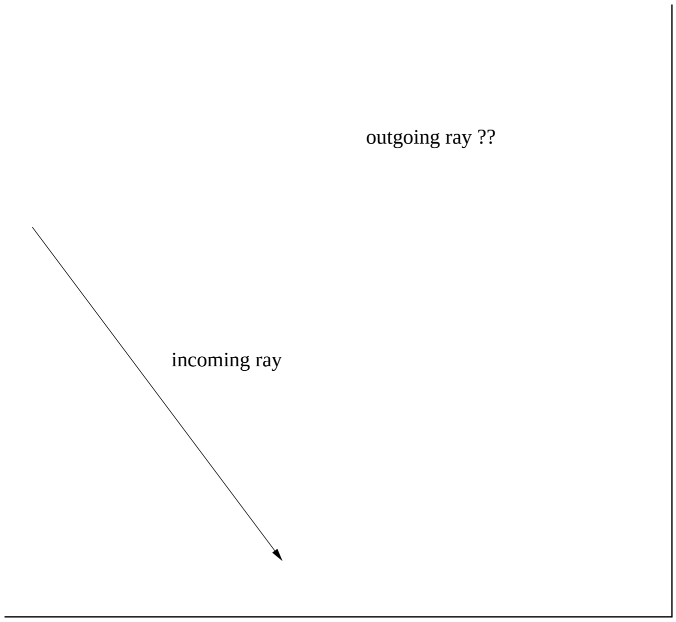
2) **Reflecting on reflection.**

   a) For a plane mirror, show that the plane formed by the incident and reflected rays is perpendicular to the plane of the mirror.
   b) Suppose you have a pair of perpendicular mirrors and that incident and reflected rays form a plane that's perpendicular to *both* mirrors. What relationship does the *final* reflected ray bear to the *initial* incident ray?
   c) Suppose you have three mutually perpendicular mirrors and an incident ray that eventually strikes all three mirrors. What relationship does the final reflected ray bear to the initial incident ray? Are there practical purposes that to which the result can be put?

      Hint: Think of the respective incident and reflected rays as vectors and keep track of *direction* as the vectors get reflected. A particularly nice choice of coordinates can help here.

3) **Constructing the minimizing hub point.** Given a triangle ABC, call a hub point H a *trisection point* if the three angles formed by AH, BH, and CH are equal.

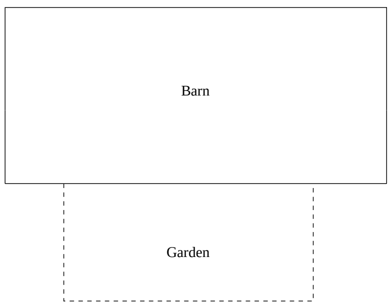
   a) Show that a triangle with an angle that's greater than $120°$ does *not* have a trisection point.



b) For a triangle with no angle greater than 120°, construct the trisection point.

Hint: Given a chord on a circle and a point P on the circle but not on the chord, what can you say about the *chord angle* at P—the angle formed by segments from P to the endpoints of the chord. If Q is on the circle but on the *other* side of the chord, how are the chord angles at P and Q related?

4) You want to fence in a *rectangular* garden one side of which is along a barn. If you have a fixed length of fence, how do you enclose the largest possible area? (Try to use what we know about the isoperimetric problem for rectangles to make a *geometric* argument.)



5) Given n numbers

$$x_1, \ldots, x_n,$$

whose sum is fixed, what values for the x_k *maximize* their product? What does this tell you about optimum shape in three dimensions? What about higher dimension?

6) You have four *straight* pieces of fence with lengths

$$1, 1, 2, 2.$$

If you enclose a polygonal region by joining these lengths of fence, what configuration maximizes the enclosed area? Note that you can order the pieces in different ways. Which ordering gives the maximum area?

Do the same for pieces of length

$$1, 2, 2, 3.$$

What are the possible configurations? Can you make a conjecture about general cases?

Hint: Try to apply the solution to the general isoperimetric problem.