Sample project topics   Math 355

Bear in mind: The quality of a project is directly related to its having **substantive content** and a **well-defined**, **narrow focus**. The following are intended to be **suggestive**. Many topics need a narrowing of focus.

Optimal paths and shapes

- Explore shortest path problems in 4-point networks in the plane. Is the solution for the 3-point network of use here?

- Explore shortest path problems in 4-point networks in three-dimensional space. Is the solution for a 2D network of use here? Reasoning by analogy, what are the various ways of connecting the points? What can you say about the optimum *network lengths*? How far can analogical reasoning take you?

- An ellipse is the set of points whose "network distance" to two fixed points is constant. Investigate sets of points in the plane whose network distance to three given points is constant. What about the analogous situation in three dimensions?

- Investigate the geometry of soap bubbles or soap films—a question related to the isoperimetric problem in 3D.

- Given an equilateral triangle and an interior point P, the sum of the distances from P to the edges is equal to the triangle's altitude. Establish this claim and show how to use it to solve the three-point shortest network problem.

- Can a circle (or triangle or rectangle) be cut by n lines into a number of regions of equal areas? If so, how many regions are formed?

Symmetry, polyhedra, and tiling

- A wallpaper pattern in the plane can be expressed as a the tiling of a torus. Explain how this works and, by describing all such tilings, classify the wallpaper patterns.

- Use the fact that the angles of a spherical triangle sum to be greater than π to describe *all* groups of rotational symmetries of the sphere that have a *finite* number of elements. Each such group corresponds to a tiling of the sphere—that is, a polyhedron. Describe these tilings.

- Find the "smallest" example of a polyhedron that have the same V, E, F values as well as the same face decomposition
$$F = \sum_{k=3}^{n} F_k.$$
Can the vertex decompositions
$$V = \sum_{k=3}^{n} V_k$$
also agree?

- Investigate the soccer ball and its relationship to the icosahedron. What polyhedron is analogous to the soccer ball but is based on the octahedron? Consider the "family" of *truncated* polyhedra to which the soccer ball belongs. Study the connection of the soccer ball to the newly-found form of carbon known as fullerene.

- Explore polyhedra with icosahedral symmetry whose faces are pentagons and hexagons. Call such a structure a PH polyhedron. How many pentagons can a PH polyhedron have? The minimal case is the dodecahedron. Another is the soccer ball (P=12, H=20). Is there a PH polyhedron with $0 < H < 20$? What about when $H > 20$?

- The *rhombic dodecahedron* is based on the octahedron and is like a regular polyhedron. Study the construction of this object and describe analogues based on the tetrahedron and icosahedron.

- Study and build a *polyhedral kaleidoscope*: a "pyramid" of mirrors that's open on each ends whose reflective properties produce the image of a polyhedron. Try this for the cube, octahedron, dodecahedron, and icosahedron. Explain the geometry.

- A *semi-regular polyhedron or plane-tiling* has faces that are regular polygons—but not all the same shape as in the regular case—and that fit together in such a way that any vertex is indistinguishable from any other vertex.. Such a polyhedron is called *vertex-regular*. Study and enumerate all such polyhedra or tilings of the plane.

- The *complete graph on five vertices* is the network of ten edges that connect five vertices in every way (a vertex is not connected to itself). Use considerations of polyhedral networks on the sphere—or something else—to show that this graph cannot lie in the plane without the edges crossing.

- Can a configuration of polygons in the plane be creased/glued along edges to form two polyhedra of different types? If so, what's the "smallest" such configuration?

- Which polyhedra can make a fair die?

- Study the question of whether a tiling of the plane can consist of polygons each of which has five-fold symmetry. Note that this issue is different from the impossible task of tiling the plane with regular pentagons.

  Source: Danzer, Grünbaum, Shephard. *Can all tiles of a tiling have five-fold symmetry?* The American Mathematical Monthly, 89, No. 8 (Oct., 1982), pp. 568-570+583-585.

- Investigate the elementary theory of paper-folding (origami).

  Source: Hull, *Project origami: Activities for exploring mathematics*

Transformations

- Determine an explicit form for *reflection in the unit circle* as a map

$$(x, y) \longrightarrow (f(x, y), g(x, y)).$$

  What should happen as the radius gets arbitrarily large?

  How would you define *reflection in the unit sphere* in three-dimensional space? What would you see when you look into a "spherical mirror" of this type? How far "behind" the mirror do you appear to be? Compare this to a flat mirror and to one made of a reflective coating on a sphere (like the underside of a spoon).

- Describe stereographic projection of a circle (a 1-dimensional sphere or "1-sphere"), 2-sphere, 3-sphere, etc. Show that stereographic projection of the the 2-sphere sends circles on the sphere to circles/lines in the plane. Where does the center of the spherical circle go? From where does the center of the circle in the plane come?

Begin by examining the simple case of stereographic projection of a 1-sphere (circle) onto a 1-plane (line). This maps 0-spheres on the circle to 0-spheres on the line. (What's a 0-sphere?) Answer the analogous questions about the respective centers of circles.

Briefly discuss stereographic projection in higher dimensions.

Source: Hilbert and Cohn-Vossen. *Geometry and the Imagination.*

Three-dimensional geometry

- Explore the three-dimensional analogue of Pascal's triangle—Pascal's pyramid. How are the entries generated? What special properties do they exhibit? Are there connections to geometric structures?

- Classify isometries in three-dimensional space. How many fixed points can a 3D isometry have? A fixed point argument analogous to the 2D case is an illuminating way to go. Conjecture how the classification of isometries goes in dimension higher than 3.

- Four mutually tangent spheres create an intriguing configuration. Study some of the properties that that this structure possesses.

  Source: Eppstein. *Tangent spheres and triangle centers.* The American Mathematical Monthly, 108, No. 1 (Jan., 2001), pp. 63-66.

- Give an analytical description of the geometric conditions that produce a rainbow.

- Examine the question of how the shape of an object affects the way it floats. In particular, how can a symmetrical object float in an asymmetrical way?

  Source: Gilbert. *How things float.* The American Mathematical Monthly, 98, No. 3 (Mar., 1991), pp. 201-216.

Four-dimensional geometry

- Build models of the 4-dimensional versions of the tetrahedron or cube—called a polytope. In what sense are these *hyperpolyhedra* regular? Describe the polytope's rotational and reflective symmetries symmetries. (Recall that a reflection in $\mathbf{R}^4$ has an $\mathbf{R}^3$ for a mirror and that a rotation has an $\mathbf{R}^2$ for an axis.) Which symmetries do/don't we see in the 3-dimensional model?

  Source: Coxeter, *Regular polytopes*

- Investigate the geometry of the a three-dimensional sphere. How do you describe it in coordinates? How is it built up out of 3D slices? How much 4D volume does it enclose? (What's meant by 'volume' here?)

- Explore higher-dimensional analogues to the *general* space-slicing problem (What's the maximum number of regions that n planes in space can produce?). What acts as a "knife?" What slicing principles should we follow? How might we approach the problem? Try to solve it.

- Develop a construction for a 4-dimensional hyper-cube. Describe the structure in terms of vertices, edges, faces, and 3-dimensional cells. Count the number of each V, E, F, and C and compute the Euler number $V - E + F - C$. Enumerate the rotational symmetries of the hyper-cube. (Recall that the axis of a rotation on 4-space is a plane.)

Curves, surfaces, knots, and the like

- Show how a Klein bottle and projective plane naturally sit inside 4-dimensional space—with no self-intersections.

- Study a knot polynomial—that of Jones or Alexander, say. Describe the construction of the polynomial and its invariance under the elementary knot (Reidemeister) moves. Use the polynomial to show that the right and left-handed trefoil knots are *not* equivalent.

  Source: Messer and Straffin. *Topology Now!*

- Each point p on a curve C that encloses a convex region has a diameterD_p—the length of the longest chord across C passing through p. A curve has *constant width* when D_p is constant as p varies over C.

  Investigate constructions for and properties of *curves with constant width.*

  Source: M. Gardner. *The Unexpected Hanging and Other Mathematical Diversions*, Ch. 18.

- Explore some of the properties of a *Geoboard.* For instance, show the relationship between the euler number, the angles, and the area of a polygon.

- Investigate the geometry of *fractals.* What's meant by the term? Discuss some of the peculiar properties that they can have. Explain the notion of fractal dimension.

  Source: Falconer. *Fractal geometry: mathematical foundations and applications.*

- Consider the motion of a billiard ball moving on a rectangular billiard table. (Equivalently, you can treat the path of a light ray that moves inside a two dimensional rectangular rectangular region that has mirrors along the boundary.) Discuss the path (called the trajectory) of the ball as a function of its initial position and direction of motion. Are there trajectories that *repeat*—that is, the ball returns to its initial position while travelling with the initial direction?

  Try looking at simple cases: where the table is square or $m \times n$ (m, n are integers). Circles and ellipses are also interesting cases.

  Source: Kinsey and Moore. *Symmetry, Shape, and Space.*

- Examine the curve on a baseball that's made along the stitches—where the two pieces of skin come together. Find a way of expressing this curve—in terms of equations or parametrically. How do you go about looking for such a description? Finding the baseball's symmetries might be useful.

- Can a wobbly four-legged table be turned so that it eventually rests on all four legs simultaneously? Investigate this phenomenon of turning a table on a variety of surfaces.

  Source: Bill Baritompa, Rainer Löwen, Burkard Polster, and Marty Ross. *Mathematical Table Turning Revisited* (`arxiv.org/abs/math.HO/0511490`).

- Position a bicycle so that one of the pedals is a its lowest point—the pedal arm is vertical. Attach a rope to the lower pedal and pull from behind the bike. Will the bike move forward, backward, or neither? The question can be treated in terms of the path that the pedal—as a point—makes in a plane—called a trochoid. Investigate the different curves that the point-pedal can produce depending on the ratios of radii of 1) the front/back gears and 2) the tire/pedal arm.

Teaching

- Discuss the pedagogical value of the "transformation" point of view in geometry. Contrast this way of looking at geometry with the "synthetic" approach—based upon proof-by-construction. Can the use of vectors or complex numbers provide help to students here? At what cost?

- Devise an activity in which students visualize (in the broadest sense) some geometric property or result. To what extent does the activity develop a general way of looking at things.

- One way to approach geometry is through *symmetry*. How can considerations of symmetry be applied at various levels of math teaching (from elementary to high school)? Develop a symmetry-related activity or lesson for some grade level. What concepts does it involve? Where might the students struggle? What's the mathematical content? Is more than one "area" of math involved?