# 2 Exercises Math 444 Fall 2009

Though these won't be collected, you should write up solutions. Try to develop concise yet thorough arguments.

### Reading

Chs. 5-8

### From the text

**Ch 6** 10
**Ch 7** 13*, 14, 15
**Ch 8** 23, 31, 36

### *Notes

**Ch 7: 13** What are the possible values for the order of $y$? For each possible order, describe a group that satisfies the given conditions.

## 2.1 Minimal conditions to determine a group

Let $X$ a set with an associative binary operation $*$ defined on it. Suppose

1) there's a left identity element $e_L$:

$$e_L * x = x \qquad \text{for all } x \in X.$$

2) every element $x$has a left inverse $\ell_x$:

$$\ell_x * x = e_L.$$

What properties would $e_L$ and $\ell_x$ have to satisfy in order for $x$ to be a group? Show that they have these properties.

*Suggestion*: To show that $e_L$ is a right identity, consider the trivial statement

$$e_L * e_L = e_L$$

and make a substitution $e_L = \ell_x * x$ where $x$ can be any element of $X$.

## 2.2 Conditions that don't determine a group

Let $X$ a set with an associative binary operation $*$ defined on it. Suppose

1) there's a left identity element $e_L$:

$$e_L * x = x \qquad \text{for all } x \in X.$$

2) every element $x$has a right inverse $r_x$:

$$x * r_x = e_L.$$

Show that these properties aren't enough to produce a group.

*Suggestion*: The simplest possible counterexample is a set with two elements.

## 2.3 Groups whose elements are sets?

Let $S$ be a set of things and let $P$ be the set of subsets of $S$. Consider the following binary operations on $P$ and determine if $(P, *)$. To show that $(P, *)$ is a group you must specify an identity element and the inverse of an arbitrary element.

1) For $A, B \in P$, let $A * B = A \cup B$.
2) For $A, B \in P$, let $A * B = A \cap B$.
3) For $A, B \in P$, let $A * B = ((S - A) \cap B) \cup (A \cap (S - B))$.

   (*Note*: $S - A$ is the *complement* of $A$: what's in $S$ but not in $A$.)

## 2.4 Direct products

Suppose $G$ and $H$ are groups.

1) Show that $(G \times H, *)$ is a group—called the *direct product of* $G$ *and* $H$ where, for $g_1, g_2 \in G$, and $h_1, h_2 \in H$,
$$(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2).$$
   Here, $g_1 g_2$ and $h_1 h_2$ refer to the multiplication in $G$ and $H$ respectively.
2) Describe the group $\mathbf{Z}_2 \times \mathbf{Z}_2$. (Elements? Order of elements? Multiplication table?) Find another group that "looks like" (has the *same structure* as) this one.
3) Describe the group $\mathbf{Z}_2 \times \mathbf{Z}_3$. (Elements? Order of elements? Multiplication table?) Find another group that looks like this one.

## 2.5 Subgroups of dihedral groups

1) Consider the symmetries $D_4$ of a square. Find all subgroups of $D_4$ and, for *each* subgroup $H$, give a figure for which $H$ is *exactly* the symmetry group.

   *Suggestion*: Modify the original square to produce the desired figure, that is, to reduce the symmetry.
2) Consider the symmetries $D_6$ of a regular hexagon. Find all subgroups of $D_6$ and, for *each* subgroup $H$, give a figure for which $H$ is *exactly* the symmetry group.

   *Suggestion*: Modify the original hexagon to produce the desired figure, that is, to reduce the symmetry.

## Presentation problems

1) **Ch 7: 13**
2) **Ch 7: 15**
3) **2.4: 1, 2**
4) **2.5: 2**