Exercises 3  Math 444  Fall 2009

Though these won’t be collected, you should write up solutions. Try to develop concise yet thorough arguments.

Reading
Chs. 9-10

From the text

Note: The text uses ‘period’ rather than ‘order’ when treating group elements.

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3.1 Groups of automorphisms

Let $G$ be a group. An isomorphism from $G$ to itself is called called an automorphism of $G$.

1) Show that the set of automorphisms of $G$ forms a group.
2) Find the group of automorphisms of $Z_4$. What familiar group is it?
3) Find the group of automorphisms of the Klein-4 group. What familiar group is it?

3.2 When squaring is an automorphism

1) Suppose that $G$ is a group and that $\phi(x) = x^2$ is an automorphism of $G$.
   a) Show that $G$ is commutative.
   b) Show that $G$ does not contain an element of even order.

2) Prove the converse of the above. That is, suppose that $G$ is commutative and that $G$ doesn’t contain an element of even order. Show that $\phi(x) = x^2$ is an isomorphism.

3.3 Symmetries of the regular tetrahedron

1) Describe the group $T$ of rotational symmetries of the regular tetrahedron. Find the subgroups of $T$. For each subgroup $S$, break the tetrahedral symmetry to obtain an object whose rotational symmetry group is exactly $S$.

2) What group $U$ do we get if we include reflective symmetries of the tetrahedron? (That is, describe and count the symmetries—find the order.) Show that $U$ is generated by reflections. Find the smallest (in terms of number) generating set of reflections that you can.

3) By labeling the edges of the tetrahedron, express each symmetry in $T$ in terms of permutations of six things. This gives a map from $T$ to the symmetric group $S_6$. Is the map a homomorphism? Is the map injective (one-to-one)? Is it an isomorphism?

4) Now, express the symmetries in $U$ as permutations of six things. This gives a map from $U$ to the symmetric group $S_6$. Is the map a homomorphism? Is the map injective (one-to-one)? Is it an isomorphism?
3.4 Symmetries of the cube

1) Describe the group $C$ of rotational symmetries of the cube. Find the subgroups of $C$.

2) What group $D$ do we get if we include reflective symmetries of the cube? (That is, describe and count the symmetries.) Show that $D$ is generated by reflections. Find the smallest (in terms of number) generating set of reflections that you can.

3) By labeling the faces 1, 2, 3, 4, 5, 6, express each symmetry in $C$ in terms of a permutation of six things. (Alternatively, consider the axes connecting the opposite face-centers.) This gives a map from $C$ to the symmetric group $S_6$. Is the map a homomorphism? Is the map injective (one-to-one)? Is it an isomorphism?

4) By labeling the pairs of opposite edges 1, 2, 3, 4, 5, 6, express each symmetry in $C$ in terms of a permutation of six things. (Alternatively, consider the axes connecting the opposite face-centers.) This gives a map from $C$ to the symmetric group $S_6$. Is the map a homomorphism? Is the map injective (one-to-one)? Is it an isomorphism?

5) By labeling the pairs of opposite faces $a, b, c$, express each symmetry in $C$ in terms of permutations of three things. (Alternatively, consider the axes connecting the opposite face-centers.) This gives a map from $C$ to the symmetric group $S_3$. Is the map a homomorphism? Is the map injective (one-to-one)? Is it an isomorphism?

6) By labeling the pairs of pairs of opposite vertices 1, 2, 3, 4, express each symmetry in $C$ in terms of permutations of four things. (Alternatively, consider the axes connecting the opposite vertices.) This gives a map from $T$ to the symmetric group $S_4$. Is the map a homomorphism? Is the map injective (one-to-one)? Is it an isomorphism?

Presentation problems

1) 3.1
2) 3.2
3) 3.3: 1, 2
4) 3.3: 3, 4