Exercises 3 Math 444 Fall 2009

Though these won't be collected, you should write up solutions. Try to develop concise yet thorough arguments.

Reading

Chs. 9-10

From the text

Note: The text uses 'period' rather than 'order' when treating gorup elements.

Ch 8	34, 35, 36
Ch 9	38, 39, 40, 41
Ch 10	46, 49

3.1 Groups of automorphisms

Let G be a group. An isomorphism from G to itself is called called an *automorphism* of G.

1) Show that the *set of automorphisms* of G forms a group.
2) Find the group of automorphisms of $\mathbf{Z}_4$. What familiar group is it?
3) Find the group of automorphisms of the Klein-4 group. What familiar group is it?

3.2 When squaring is an automorphism

1) Suppose that G is a group and that $\phi(x) = x^2$ is an automorphism of G.

   a) Show that G is commutative.
   b) Show that G does *not* contain an element of even order.

2) Prove the converse of the above. That is, suppose that G is commutative and that G doesn't contain an element of even order. Show that $\phi(x) = x^2$ is an isomorphism.

3.3 Symmetries of the regular tetrahedron

1) Describe the group T of *rotational* symmetries of the regular tetrahedron. Find the subgroups of T. For each subgroup S, break the tetrahedral symmetry to obtain an object whose rotational symmetry group is *exactly* S.
2) What group U do we get if we include reflective symmetries of the tetrahedron? (That is, describe and count the symmetries—find the order.) Show that U is generated by reflections. Find the smallest (in terms of number) generating set of reflections that you can.
3) By labeling the edges of the tetrahedron, express each symmetry in T in terms of permutations of six things. This gives a map from T to the symmetric group S_6. Is the map a homomorphism? Is the map injective (one-to-one)? Is it an isomorphism?
4) Now, express the symmetries in U as permutations of six things. This gives a map from U to the symmetric group S_6. Is the map a homomorphism? Is the map injective (one-to-one)? Is it an isomorphism?

3.4 Symmetries of the cube

1) Describe the group C of *rotational* symmetries of the cube. Find the subgroups of C.
2) What group D do we get if we include reflective symmetries of the cube? (That is, describe and count the symmetries.) Show that D is generated by reflections. Find the smallest (in terms of number) generating set of reflections that you can.
3) By labeling the faces $1, 2, 3, 4, 5, 6$, express each symmetry in C in terms of a permutation of six things. (Alternatively, consider the *axes* connecting the opposite face-centers.) This gives a map from C to the symmetric group S_6. Is the map a homomorphism? Is the map injective (one-to-one)? Is it an isomorphism?
4) By labeling the *pairs* of opposite edges $1, 2, 3, 4, 5, 6$, express each symmetry in C in terms of a permutation of six things. (Alternatively, consider the *axes* connecting the opposite face-centers.) This gives a map from C to the symmetric group S_6. Is the map a homomorphism? Is the map injective (one-to-one)? Is it an isomorphism?

   Compare the two cases
5) By labeling the pairs of opposite faces a, b, c, express each symmetry in C in terms of permutations of three things. (Alternatively, consider the *axes* connecting the opposite face-centers.) This gives a map from C to the symmetric group S_3. Is the map a homomorphism? Is the map injective (one-to-one)? Is it an isomorphism?
6) By labeling the pairs of pairs of opposite vertices $1, 2, 3, 4$, express each symmetry in C in terms of permutations of four things. (Alternatively, consider the axes connecting the opposite vertices.) This gives a map from T to the symmetric group S_4. Is the map a homomorphism? Is the map injective (one-to-one)? Is it an isomorphism?

Presentation problems

1) **3.1**
2) **3.2**
3) **3.3: 1, 2**
4) **3.3: 3, 4**