Exercises 4 Math 444 Fall 2009

Though these won't be collected, you should write up solutions. Try to develop concise yet thorough arguments.

Reading

Groups and their Graphs: Chs. 11, 13

From the text

Ch 11 52, 57

4.1 A finite group as a group of permutations

Cayley's permutation theorem says that any finite group G is isomorphic to a subgroup of the symmetric group $S_{|G|}$.

a) For the Klein-4 group V, show explicitly an isomorphic correspondence between V and a subgroup of S_4.
b) For the dihedral group D_3, show explicitly an isomorphic correspondence between D_3 and a subgroup of S_6.

4.2 Improving on Cayley's theorem

Sometimes we can do better than the correspondence that establishes Cayley's theorem: when G is isomorphic to a subgroup of S_n where $n < |G|$.

a) Is the Klein-4 group V isomorphic to a subgroup of S_n for some $n < 4$?
b) We've seen that $D_3 \simeq S_3$. What about D_4? Is it isomorphic to a subgroup of S_n for some $n < 8$? If so, what's the smallest such n for which this happens? Is it isomorphic to a subgroup of the alternating group A_n?
c) Consider the tetrahedral rotation group T. Is it isomorphic to a subgroup of S_n for some $n < 12$? If so, what's the smallest such n for which this happens? Is it isomorphic to a subgroup of the alternating group A_n?

4.3 A normal subgroup of the tetrahedral group

a) Use the geometry of the group T of rotational symmetries of the regular tetrahedron to find a normal subgroup N of T.

   Suggestion: Consider the way that T permutes the axes of rotation and look for a set of rotations that generate a one-of-a-kind subgroup.
b) What's the quotient group T/N?
c) By consideration of the way T permutes axes of rotation, find a map ϕ from T to a symmetric group for which $N = \ker \phi$.

4.4 Numerical quotient groups

a) The real numbers $\mathbf{R}$ form a commutative group under addition so that the integers $\mathbf{Z}$ form a normal subgroup of $\mathbf{R}$. Describe the equivalence relation on $\mathbf{R}$ associated with the quotient group $\mathbf{R}/\mathbf{Z}$. (Recall that G/H is the set of cosets of H, which partition G.) Give a geometric description of $\mathbf{R}/\mathbf{Z}$.

b) Let $n\mathbf{Z}$ be the integers that are multiples of n Show that $n\mathbf{Z}$ is a normal subgroup of $\mathbf{Z}$. Describe the quotient $\mathbf{Z}/n\mathbf{Z}$.

4.5 The determinant as a homomorphism

The set $\mathrm{GL}_n(\mathbf{R})$ of invertible matrices with real entries or linear transformations on $\mathbf{R}^2$ forms a group. The determinant of a matrix or linear transformation can be thought of as a map

$$\det : \mathrm{GL}_n(\mathbf{R}) \longrightarrow \mathbf{R}^*$$

where $\mathbf{R}^*$ is the group $\mathbf{R} - 0$ under multiplication. Since $\det AB = \det A \det B$, det is a homomorphism.

a) When $n = 2$, what's $\ker(\det)$. Give a geometric description of the quotient $\mathrm{GL}_2(\mathbf{R})/\ker(\det)$. (You can think of 2×2 matrices as vectors in $\mathbf{R}^4$.)

b) Show that the set L of matrices with the form

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \qquad \text{where } \det M \neq 0$$

is a subgroup of GL_2. Give a geometric description of the quotient $L/\ker(\det)$. (Of course, we're thinking of det as map from L to $\mathbf{R}^*$.)

Presentation problems

a) **4.1: a, 4.2: a**

b) **4.1: b, 4.2: b**

c) **4.4**

d) **4.5**