Exam 2 Math 444 Fall 2009
Due: 7P, Monday, 7 December

- Write concisely and clearly.
- Work in a group of two or three. Submit one paper for the group.
- *Ethic*: In completing this paper you can consult the text, other members of your group, and the instructor. The written work that you submit should be entirely the product of the group collaboration.

1 Groups of order six

Describe *all* groups of order six. (Two groups are equivalent if they're isomorphic.) Make sure that your approach is sufficiently systematic to establish your list as *complete*. Make your arguments concise. *Suggestion*: Explore what it takes to construct group graphs.

2 Groups of order p^2 where p is prime

Let p be a prime number and let G be a group with $|G| = p^2$. The task is to show that G is either cyclic or isomorphic to $\mathbf{Z}_p \times \mathbf{Z}_p$. So, suppose that G is not cyclic.

a) Let $a \neq e$ be some element of G and $A = \langle a \rangle$. What's the order of A?

b) Consider the cosets of A:
$$G/A = \{A, g_2A, \ldots, g_nA\}.$$
What's the value of n?

c) Let $\phi : G \to S_n$ (S_n is the symmetric group) be defined by
$$\phi(x)(g_kA) = xg_kA.$$
Note that $\phi(x)$ is a permutation on G/A. First, show that ϕ is a homomorphism. Now, prove that the only two possibilities for the kernel of ϕ are 1) $\ker\phi = A$ or 2) $\ker\phi = \{e\}$. But, case 2) implies that ϕ is injective. Consider the order of the image $\phi(G)$ to show that this can't happen.

Now conclude that A is a normal subgroup of G.

d) Let $b \in G - A$. What's the order of b? Since A is normal,
$$bab^{-1} = a^k \quad \text{for some } k,\ 1 \leq k < p.$$
Establish
$$a^{-1}ba = b^m \quad \text{for some } m,\ 1 \leq m < p.$$

e) Derive
$$b^{m-1} = a^{k-1}$$
and deduce
$$b^{m-1} = a^{k-1} = e \qquad \text{and} \qquad ab = ba.$$

f) By considering *all* of the products a^ib^j show that $G \simeq \mathbf{Z}_p \times \mathbf{Z}_p$. Describe the isomorphism explicitly.