Sample project topics Math 444

Bear in mind: The quality of a project is directly related to its having **substantive content** and a **well-defined**, **narrow focus**. The following are intended to be **suggestive**. Many topics need a narrowing of focus. Computer graphics and experiments might help to illuminate some of the ideas and results.

- Classify the finite subgroups of the group of isometries of the plane.
- Classify wallpaper patterns—that is, tiling patterns that have two independent translational symmetries.

  Source: M. A. Armstrong, *Groups and Symmetry*, Chs. 25, 26

  B. Grünbaum, G. C. Shephard, *Tilings and patterns*
- Prove the "Sylow Theorem" stated below.

  Let G be a group with $|G| = p^m q$ where p is prime and does not divide q. Then G contains at least one subgroup of order p^m.

  Source: M. A. Armstrong, *Groups and Symmetry*, Chs. 20
- The set of "motions" of a sphere is an infinite group called O_3—3×3 matrices where $AA^T = I$ and $\det A = \pm 1$. Classify the *finite* subgroups of O_3.

  Source: M. Senechal, *Finding the Finite Groups of Symmetries of the Sphere*, The American Mathematical Monthly, 97, No. 4 (Apr 1990), pp. 329-335.
- Investigate the "symmetry principle" and apply it to determine the effective resistance between a pair of vertices of a cube whose edges are resistors.

  Source: J. Rosen, *Symmetry Discovered*, Ch. 6 and *A Symmetry Primer for Sceintists*, Ch. 4.
- Explore some aspect of the application of group theory to chemistry.

  Source: I. Hargittai, M. Hargittai, *Symmetry through the eyes of a chemist*

  S. Bhagavantam, *Crystal symmetry and physical properties*
- Find all groups of order 8 (up to isomorphism).
- Find all groups of order 12 (up to isomorphism).
- Find all finite subgroups of the complex numbers under multiplication.
- Use mobius transformations to express the group of symmetries of the regular tetrahedron T. Find three special polynomials that are invariant under this group. That is, a polynomial $P(z)$ such that

  $$P(Az) = P(z) \quad \text{for all } A \in T.$$
- Describe and count the rotational symmetries of the icosahedron (dodecahedron). Call this the *icosahedral group* I. Find five tetrahedra each of which is preserved by a different "tetrahedral group" in I. Show that I permutes these five tetrahedra and that each permutation is even. Conclude that I is isomorphic to the alternating group A_5.

Using the six pairs of antipodal vertices, express I in terms of a subgroup of S_6—as permutations of six things. Exhibit the correspondence between the permutations of five tetrahedra and six pairs of antipodal vertices. This shows that A_5 can be mapped into S_6 in a way that's not straightforward—that is, by fixing one thing.

- Consider the group $G = \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. Describe the subgroups of G. Each non-identity element has order two. An automorphism permutes the seven elements of order two. How many automorphisms of G are there?

  A set of seven points ('point' in an abstract sense) can be taken three at a time to form a *finite geometry* S of seven points and seven lines. (A line is just a triple of points.) A *collineation* of S is a transformation of S that preserves the lines in S. The collineations of S form a group. Compare the group of automorphisms of G to the group of collineations and show that they are isomorphic.

- The imaginary number i is a square root of -1 and so has order 4 under multiplication. Show that a noncommutative group Q of order 8 is generated by i and another square root j of -1, where $j \neq i^3$. Find all the subgroups of Q and show that each one is normal.

  Source: *Groups and their graphs*, Ch. 12.

- How many *essentially different* labelings—say what this means—are there of the six pairs of antipodal faces of the regular dodecahedron.

- Find an interesting piece of theory to develop or problem to solve.