Assignment 2 Math 451 Spring 2009
Due: 12N Friday 17 April

Reading

> ***Curved Spaces*:** 2.1-2.3, 2.5, 2.8

Exercises

Write in concise, clear *sentences* (incorporating symbolic notation and computations).

1) Use the Gauss map to find the Gaussian curvature of a sphere of radius R at any point.

   Now find the Gaussian curvature of a cylinder of radius R—infinitely long—at any point.

2) Describe the Gauss map for a torus. Where is the Gaussian curvature positive, negative, and zero?

   The total (or integral) curvature of a surface S is given by

   $$\int_S k(x)\, dA(x)$$

   where $k(x)$ is the Gaussian curvature at x and $dA(x)$ is an element of area at x.

   Without doing an explicit calculation, use the Gauss map to compute the total curvature of a torus.

3) Find the Gaussian curvature of a cone with vertex angle α at any point other than the vertex. (The *vertex angle* is the angle formed by the lines of intersection between the cone and a plane through the cone's axis.) What's the total curvature of the cone including the vertex? Where does the curvature "live?"

   Cut the cone from the edge to the vertex and lay it flat. The amount of missing angle is the cone's angle defect. Find this defect in terms of α and observe that it equals the Gaussian curvature.

Class presentations

1) ***Curved Spaces*:** 2.6
2) Let $\gamma(s)$ be a curve in $\mathbf{R}^3$ parametrized by arc length with curvature and torsion $\kappa, \tau > 0$ and (T, N, B) its Frenet frame. Set

$$\widetilde{\gamma}(s) = \int_0^s B(s)\, ds.$$

   Show that $\widetilde{\gamma}(s)$ is also parametrized by arc length.

   Let $\widetilde{\kappa}$, $\widetilde{\tau}$, and $(\widetilde{T}, \widetilde{N}, \widetilde{B})$ be the curvature, torsion, and Frenet frame for $\widetilde{\gamma}$. Express the curvature, torsion, and Frenet vectors for $\widetilde{\gamma}$ in terms of the curvature, torsion, and Frenet vectors for γ.
3) Let $\gamma(s)$ be a curve in $\mathbf{R}^3$ parametrized by arc length with $T(s)$ its tangent vector field. Consider the path on the unit sphere made by $T(s)$ and set $\sigma(s)$ equal to the path's arc length. Derive

$$\frac{d\sigma}{ds} = \kappa(s).$$

4) For a space curve parametrized by arc length s, express the equation of the osculating plane as a function of s. Show that the curve is planar if and only if there is some point $P \neq 0$ that belongs to every osculating plane.