Math 451   Spring 2009
Lab 1: Path of centers of curvature

Reminders

- Save your work *every time* you enter new material in a cell and *before* you send it to the kernel for evaluation.
- Your submission—a *single* worksheet called *yourname_lab#.mw*—should contain responses to the **Worksheet exercises**. Every cell of your notebook should have, at least, a preceding comment that describes the cell's content and what the outcome means.
- Remember the `Maple Help` facility accessible in the `Help` menu.
- *Maple* code appears in `typewriter font`.

Tasks

Write a program that will produce the path of the centers of curvature for a given curve. Investigate the relationships between the two curves.

A curvature formula

Suppose Γ is a curve in $\mathbf{R}^2$ parametrized by the map $\gamma = (x(t), y(t))$ on $[a, b]$ (not necessarily by arc-length). By a straightforward, albeit tedious, calculation we can express the curvature *vector* at $\gamma(t)$ by

$$k(t) = \frac{\ddot{x}\,\dot{y} - \dot{x}\,\ddot{y}}{(\dot{x}^2 + \dot{y}^2)^2}\,(\dot{y}, -\dot{x}) \quad \text{where } \dot{u} = \frac{du}{dt}.$$

Recall that the center of curvature of Γ at $\gamma(t)$ is the center of the osculating circle at $\gamma(t)$. This point is given by

$$C_\gamma(t) = \gamma(t) + \frac{1}{\|k(t)\|}\,N$$

where N is the unit normal in the direction of the curvature vector.

Plotting a curve

You can represent a curve given by $g = (x(t), y(t))$ in *Maple* as a map

$$g{:=}t{-}{>}(x(t),y(t)).$$

(The symbol 'f' is used for simplicity.) The command

$$plotname{:=}[g(t),\ t{=}a{.}{.}b]$$

creates a *plot structure* (the set of data) for plotting the curve over $[a, b]$. To actually plot the curve, you can use

<div align="center">

`plot(plotname).`

</div>

To plot more than one curve, use

<div align="center">

`[plotname1,plotname2].`

</div>

Note: Before *Maple* can recognize these graphics objects, you need to load the plotting package. At the *beginning* of the worksheet, running

<div align="center">

`with(plots)`

</div>

will load the package.

Suggestion: Try to make your code *robust* in the sense that it can be applied to any curve. To make specific plots (or to apply other procedures, for that matter), you simply change the input map for the curve and the output is the desired plot (or whatever you're trying to obtain).

The cycloid

If a circle rolls along a line, the path that one point on the circle follows as the circle rolls is called a cycloid. In Figure 1, the circle on the left rolls to the right and the original point of contact traces a cycloid. Using the indicated angle t as a parameter, express the cycloid curve as

$$\gamma(t) = (x(t), y(t)).$$

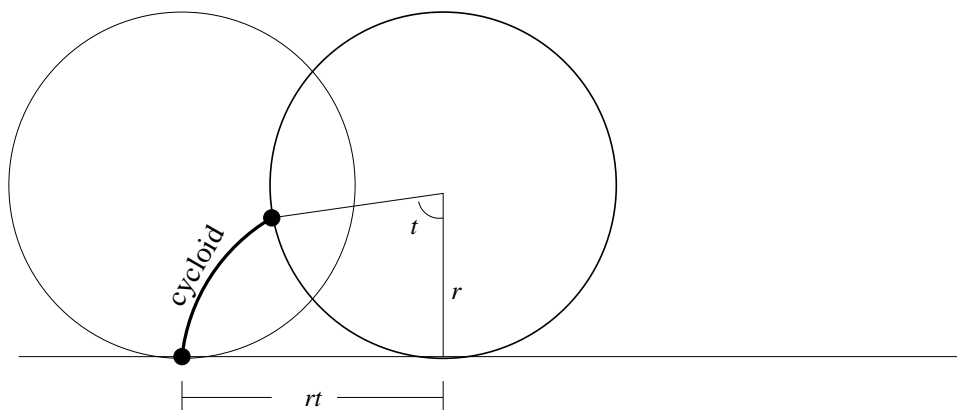*Suggestion*: The original point of contact is a good choice for an origin.

Figure 1: The cycloid

Worksheet exercises

For each of the following curves, plot both the curve and its path of centers of curvature.

Some technical issues. The kth derivative of a function defined by

$$f:=x->f(x)$$

is specified by

$$diff(f(x),x\$k).$$

Maple sees the result of this as an *expression* not a function. You can turn an expression into a function that you can evaluate with

$$function\_name := a->subs(x=a,expression).$$

For a vector $v = (x, y)$ (such as $g(t)$), you can extract the components with

$$v[1] \;(=x) \qquad v[2] \;(=y).$$

In your procedure for $C_\gamma(t)$, you need to input $\gamma(t)$. If you want *Maple* to "see" $g(t)$ as a vector whose components can be extracted, you can enclose it in square brackets. So, to deal with this annoying detail, you can use $[g(t)]$ as input rather than $g(t)$.

1) $\gamma(t) = (\cos t, 2 \sin t) \quad t \in [0, 2\pi]$
2) the graph of $y = x^2 \quad$ for $x \in [-2, 2]$
3) the cycloid path for a circle of radius 1 and one revolution.

Bonus. For the cycloid, what curve does the the path of centers of curvature turn out to be? Make a change of coordinates to show it.

Connecting a curve with its centers of curvature

Now, we want to draw a line segment between $\gamma(t)$ and $C_\gamma(t)$. (Note that the parameter value is the same.) A simple way to do this is with **pointplot**. Say you want to connect $(1, 2)$ and $(3, 4)$, you can use

$$pointplot([[1,2],[3,4]], connect=true).$$

Note the two sets of square brackets; the outermost set indicates that you have a list of something while the innermost ones enclose the coordinates of the points. The expression **connect=true** is an option that connects the points with a line segment. The result of **pointplot** is again a plot structure that can be displayed.

Worksheet exercises

For each of the curves you plotted before, connect $\gamma(t)$ and $C_\gamma(t)$ for at least five values of t. Display these segments along with the plots of $\gamma(t)$ and $C_\gamma(t)$ that you made. To do this you can use

```
plots[display]([plotname1,plotname2], pointplotname],scaling=constrained)
```

The scaling option displays the objects in a one-to-one aspect ratio. Make sure you have this set.

In light of the graphical output, form a conjecture on the relationship between the $\gamma(t)$ and $C_\gamma(t)$. Give an analytic proof of your conjecture.

If you'd like to produce an automated procedure that produces the connecting line segments—say, where you specify the number of connecting segments and increment t by the appropriate amount, the list-building command `seq` could be useful.