“Straight” and “Angle” on Non-Planar Surfaces: Non-Euclidean Geometry Introduction

A module in the Algebra Project high school curriculum

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Notes to teacher:

The idea is not to do technical details of non-Euclidean geometry, but rather to apply the meanings of “straight” and “angle” that the students have already studied in previous work.

This section can be started at any time after Straight & Symmetries sections SS-1 and SS-2 have been covered and before SS4. The idea when studying plane geometry is to be able to ask: “What happens on the sphere?” (particularly, when it helps the understanding of plane geometry).

The teacher (and interested students) can find more information and activities about non-Euclidean geometry on the website: http://www.math.cornell.edu/~mec/mircea.html

In activities in **NE1**, **t**he students are divided into groups of 3 or 4 and each group should be supplied with Styrofoam balls (at least 4 inches in diameter), or a beach ball or similar smooth sphere – soccer balls or volleyballs will work, but basketballs tend not to be smooth enough. Each group should also have push pins (if using Styrofoam), some lengths of ribbon (long enough to fit around the sphere and be tied), toys cars (with non-steerable wheels), yarn (long enough to fit around the sphere and be tied), and a transparent disk (about 2 inches cut from a transparency -- small enough to nearly lay flat on the sphere) with a small hole punched in the center with a straight line segment drawn on it thru the center from edge to edge.

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Mathematician: In the Straight & Symmetries sections **SS1** and **SS2** we explored meanings of “straight” on the plane. We agreed that:

1. A straight line has no turning left or right at any point.
2. A straight line contains the shortest distance between any two of its points.
3. A straight line has mirror symmetry (left side and right side are the same).
4. A straight line has half-turn (180°) symmetry about every point on the line.
5. A straight line has translation symmetry along itself.

We will now apply these notions of “straight” to other surfaces.

NE1. Spheres.

Dialogue NE1-D1.

Mathematician: Imagine yourself to be a bug crawling around on a sphere. This bug can neither fly nor burrow into the sphere. The bug’s universe is just the surface; it never leaves it.

Student 3: Oh, you mean like an ant crawling on a beach ball?

Mathematician: Yes, that’s right. And notice that the ant appears not to be affected by gravity or anything off the surface.

Student 3: Yeah, ants seem perfectly happy walking upside down on the bottom of a ball.

Mathematician: Now imagine you are this bug (or ant) on the surface of a sphere. What would you experience as “straight”? How can you convince yourself of this? Use the meanings of straight that we agreed on in Section I.

Student 2: But wouldn’t a bug only experience straight if it burrowed thru a straight tunnel?

Mathematician: Yes, this would probably be true if the bug was thinking of navigating in the 3-d space that contains the sphere; but imagine that the bug is bound to the surface of the sphere, much the same way that we are bound to the surface of the earth.

The important thing here is to **think in terms of the surface of the sphere, not the solid 3-dimensional ball**. Clearly, to us outside of the sphere any path on the sphere will appear curved (not-straight). But what will be the experience of the bug who is only experiencing the surface of the sphere?

Note: *A good example of how this type of thinking works is to look at an insect called a water strider. The water strider walks on the surface of a pond and has a very 2-dimensional perception of the world around it — to the water strider, there is no up or down; its whole world consists of the 2-dimensional plane of the water. The water strider is very sensitive to motion and vibration on the water's surface, but it can be approached from above or below without its knowledge. Hungry birds and fish take advantage of this fact. This is the type of thinking needed to visualize adequately properties of straight lines on the sphere. For more discussion of water striders and other animals with their own varieties of intrinsic observations, see the delightful book "The View from the Oak", by Judith and Herbert Kohl.*

NE1-1. Group Activity

Imagine yourself to be a bug crawling around on a sphere. (This bug can neither fly nor burrow into the sphere.) The bug's universe is just the surface; it never leaves it. What is "straight" for this bug? What will the bug see or experience as straight? How can you convince yourself of this? Think about your experiences with straight lines on the plane.
What would angles be like for the bug?
How would the bug experience perpendicular lines, triangles, squares, circles, and so forth?
What in the bug's experience would be like being on the plane and what would be different?

Each group reports their findings to the whole class, which leads to class discussion.

NE1-2. Worksheet (Group): Now each GROUP explores the following activities and questions using their beachball, yarn, ribbons, and transparent disk. They then come up with some answers that are later shared with the whole class. They report back to the whole class after each mini-activity or at the end. The teacher should make this call depending on the students in the class and the length of time in a class period. In each case connect what you do here with what you did in Straight & Symmetry, **SS1**, and **SS2**.

A. Mark two points on the sphere. Stretch a piece of yarn between the two points.

What do you notice? Where does it seem to fit best?	**Related to which of the 5 properties of straightness?**

B. Take a ribbon and try to "lay it flat" on the sphere.

What do you notice? Along which paths does the ribbon lie flat?	**Related to which of the 5 properties of straightness**
What does this have to do with mirror symmetry?	**Related to which of the 5 properties of straightness?**

C. Imagine the bug walking along a path on the sphere.

On what paths will the bug's right and left legs be moving at the same speed?	Related to which of the 5 properties of straightness?

D. Take a small toy car with its wheels fixed to parallel axes so that, on a plane, it rolls along a straight line. Try rolling this toy car on the sphere.

What do you notice? What curves does the car follow?	Related to which of the 5 properties of straightness?

E. Use your transparent disk to explore the symmetries of curves on the sphere in the same way you did in section SS-2. {Your teacher can help remind you of these activities from **SS-2.**

What do you notice? Which curves have symmetry?	Related to which of the 5 properties of straightness?

Whole class discussion. Each group reports back to the whole class about what they observed and the whole class agrees on statements about "straight" on a sphere.

Dialogue NE1-D2.

Mathematician: What did you observe?

Student 1: The biggest circles on the sphere (the ones that divide the sphere in half) seem to satisfy all of the same five properties that we talked about for straight lines on the plane.

Student 2: And smaller circles (ones that don't divide the sphere in half) seem to not satisfy these five properties.

Mathematician: Good. Mathematicians call the first circles **great circles** (dictionary term) and define them by saying: Great circles are those circles on a sphere that are the intersection of the sphere with a plane through the center of the sphere.

Student 3: But I thought that the bug can't see such planes because they would be off the surface.

Mathematician: That is correct. This description of great circles is for us who are viewing the great circles on the sphere from the outside. How does the bug view the great circles?

Student 1: As a straight line! The bug would think it is going straight.

Mathematician: Mathematicians call our point-of-view from the outside "**extrinsic**" and the bug's view of only the surface as "**intrinsic**". So we say that the great circles on a sphere are *extrinsically* (from our outside point-of-view) *curved* but they are *intrinsically* (from the bug point-of-view) *straight.*

NE1-3. Individual writing

Write a paragraph (or more) about: Why are you convinced that the great circles on a sphere are intrinsically straight. Or, if you are not convinced, then what is unanswered for you? Take what you learned from **NE1-1** and **NE1-2** (and/or other things) and write what is most convincing to you. Do you think other circles on the sphere can be straight? Why?

Each student's writing is shared with the class (by reading and/or posting?) and students make constructive comments on each other's papers. This leads to a whole class discussion about which paths on the globe are intrinsically straight and which are not.

NE1-4. Whole class discussion. Locate a globe of the earth in the classroom so all can see it.

Mathematician: We have been exploring why great circles are intrinsically straight and why they are the paths that a bug would experience as straight.

Student 3: This is like the equator on the earth – this must be a great circle.

Mathematician: Correct. Can you show the other students this on a globe? Do you see other circles on the globe that are great circles? Paths that are intrinsically straight on a sphere (or other surfaces) are called ***geodesics*** **(geo-DEES-iks).** (dictionary term).

Student 1: So on the sphere, arcs of great circles are geodesics.

Student 2: The arcs that go from the North Pole to the South Pole are also geodesics.

Mathematician: Yes, and these arcs that go from Pole to Pole are called Longitude lines on the globe. Are there other geodesics on the globe.

Student 3: Well, any circle that divides the sphere in half is a geodesic.

Mathematician: Why is this important for navigating airplanes and ocean-going ships**?**

Student 1: Ships and airplanes want to go the shortest routes.

Mathematician: Find the shortest path on the Earth from your city to Cairo, Egypt. Then try this for other pairs of cities.

Here is what we have agreed on:

On a sphere:
- Great circles are the intrinsically straight (geodesics) paths and other circles are not.
- A great circle contains the shortest distance between any two of its points.
- A great circle has mirror symmetry.
- A great circle has half-turn symmetry about every point on it.
- A great circle has translation symmetry along itself.

Mathematician: It is natural for you to have some difficulty experiencing straightness on the sphere; it is likely that you will start out looking at spheres and the curves on spheres as 3-d objects.

Student 2: Yeah. They look three-dimensional!

Mathematician: However, intrinsically, from the bug point-of-view they are two-dimensional because as the bug walks along the surface of a sphere it has only two choices of direction -- left or right and forward or back.

> Mathematician: Notice that, on a sphere, straight lines are intrinsic circles (points on the surface situated at a fixed distance from a given point on the surface) — special circles whose circumferences are straight! Note that the equator on the earth is an intrinsic circle with two intrinsic centers: the North Pole and the South Pole. In fact, any circle on a sphere has two intrinsic centers!

NE2. Angles and Figures on a Sphere:

NE2-1. Optional Group Activity. Review what you explored in SS3: Angles & symmetries.

Use your yarn to designate two intersecting intrinsic straight line segments on the sphere. Use your transparent disk and yarn as appropriate to explore the symmetries of the intersecting lines as you did in **SS3-1**. What do you find? How is it different on a sphere?
Reread the dialog **SS3-D1** about the Vertical Angle Theorem. Look at the proof of the VAT using half-turn symmetry of lines. Does this apply on a sphere? Explain.
Look at the proof of VAT using mirror symmetry and angle bisector. Does this apply on a sphere? Explain.

Each group reports back to the whole class. The class discusses and agrees on statements.

NE2-2. Group activity. (Each group can pick which of these they want to start on.)
Use your yarn to mark intrinsically straight line segments on the sphere holding the ends with tape. Compare what you see with what happens with straight lines on the plane. Write down what you notice and how it compares with straight lines on the plane.

Draw a large triangle on the sphere	**What do you notice about the angles?**
Draw two lines that are parallel transports along a transversal	**What do you notice?**
Try to draw a square on the sphere.	**How does it compare to figures on plane?**
Try to draw other figures	**How does it compare to figures on plane?**

Each group reports back to the whole class. The class discusses and agrees on statements.

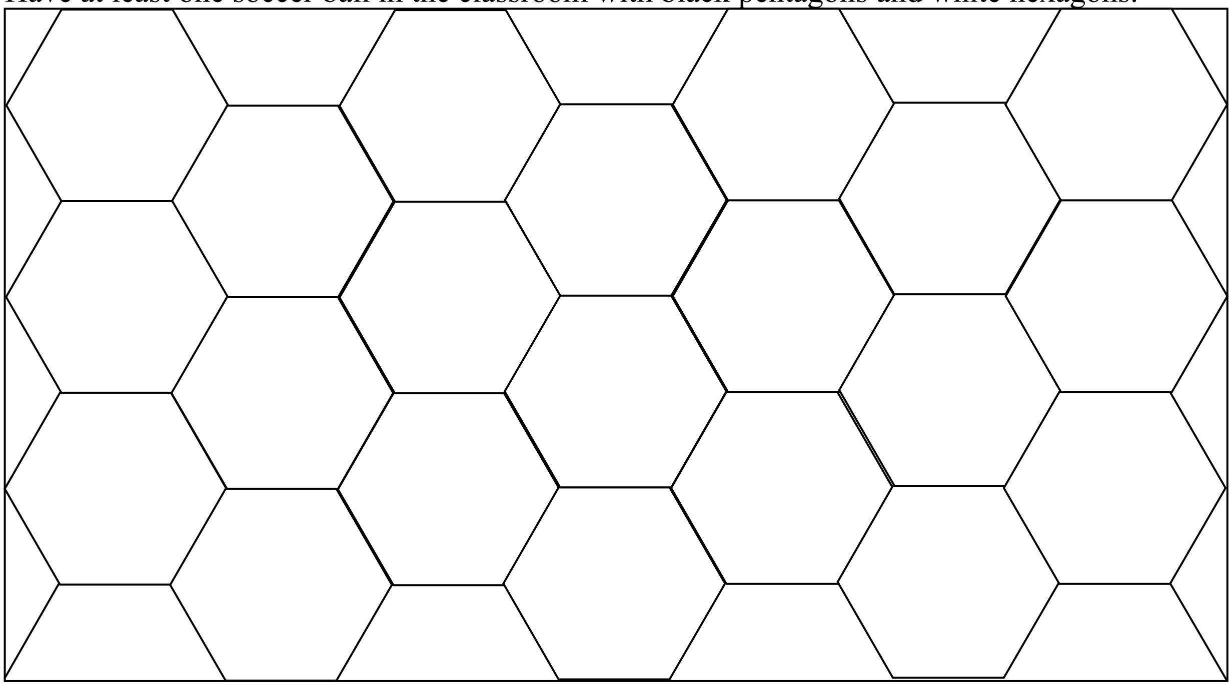
{Note to teacher}: Here is a list of some of the statements that students have come up with in other schools:

1. Any two great circles intersect, but parallel straight lines on the plane do not intersect.
2. Great circles are finite and come back on themselves, but straight lines go on forever on the plane.
3. Even though great circles are finite you can walk along the great circle forever in either direction -- this is the same as for straight lines on the plane except that on the sphere you are retracing your steps.
4. The Vertical Angle Theorem holds, the same as on the plane.
5. Some great circles are parallel transports along a transversal (another great circle). Example: The longitude lines on the earth are parallel transports of each other along the equator. On the plane parallel transported lines never intersect.
6. Three great circles can form a triangle (in fact 8 triangles!).
7. Some triangles on the sphere have angle sum more than 180 degrees. Example: there is a triangle with three right angles. On the plane the sum of the angles of a triangle is always 180 degrees.
8. Not all arcs in a great circle are the shortest distance between their endpoints.
9. There are “square-like” figures on the sphere with 4 equal sides and 4 equal angles.
10. There are “almost squares” on the sphere with 3 right angles and 2 equal sides, or 2 right angles and 3 equal sides.

It is not important that the students come up with or agree on all of these statements. It is also OK if they disagree with each other or if they make conjectures/questions that no one knows how to prove (for example: Is there any triangle on the sphere which has angle sum equal to 180 degrees?). Disagreements and open conjectures and questions can be a good source for individual or small group projects and papers. It would be a profitable experience if the students debated with each other (orally and/or in writing).}

NE3. Hyperbolic Soccer Balls.

Have at least one soccer ball in the classroom with black pentagons and white hexagons.



Mathematician: As you see, hexagons (six sides) can fill up the plane. This is called a **tessellation** of the plane. Note that in this tessellation every hexagon is surrounded by six other hexagons. Could we instead have a tessellation with pentagons (five sides) each surrounded by 5 hexagons?

Student 1: Sure that looks like a soccer ball!

Mathematician: Yes, let's look at the soccer ball and see that each black pentagon is surrounded by five white hexagons.

Student 2: I think we could build one out of paper.

Mathematician: Now, what would happen if we started with a seven-sided polygon, called a *heptagon* (dictionary term) and surrounded it with seven hexagons?

Student 1: I can't imagine it. It would be weird.

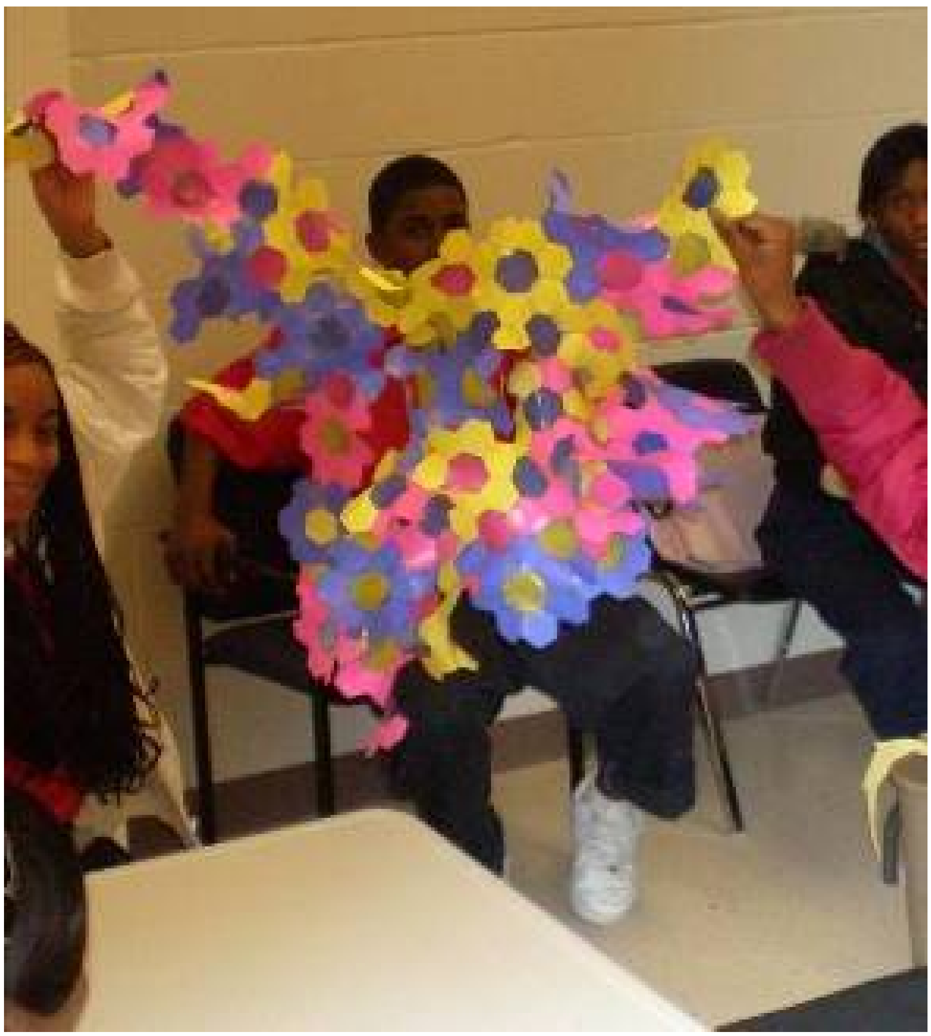
Student 2: Can we try to build one?

Mathematician: Yes, let's do that. We will both build ones with pentagons and one with heptagons.

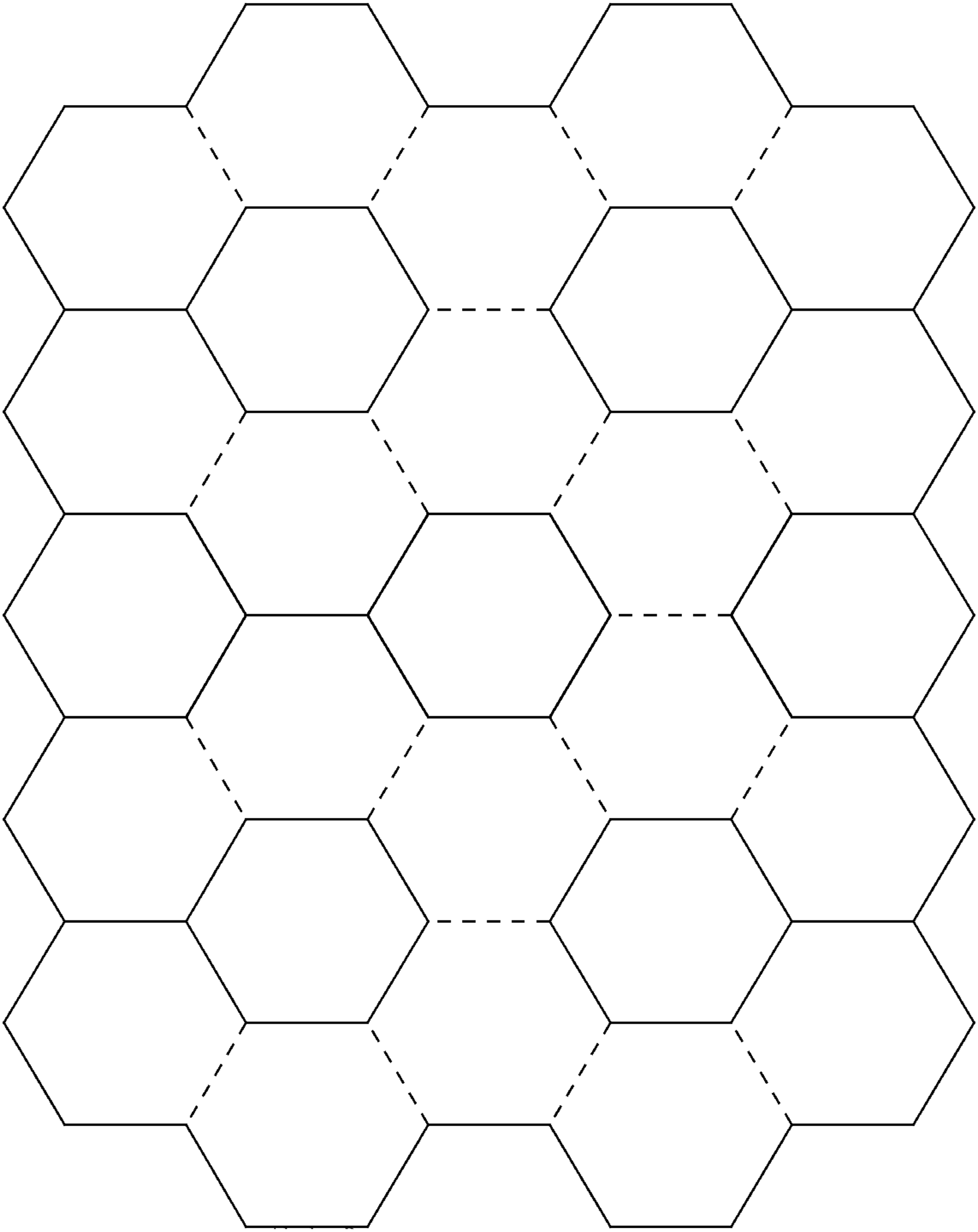
NE3-1. Activity (Individual/Group)
Each group is given sheets of templates (next page) – one for each student. The students will cut out the hexagons and heptagons according to the directions:

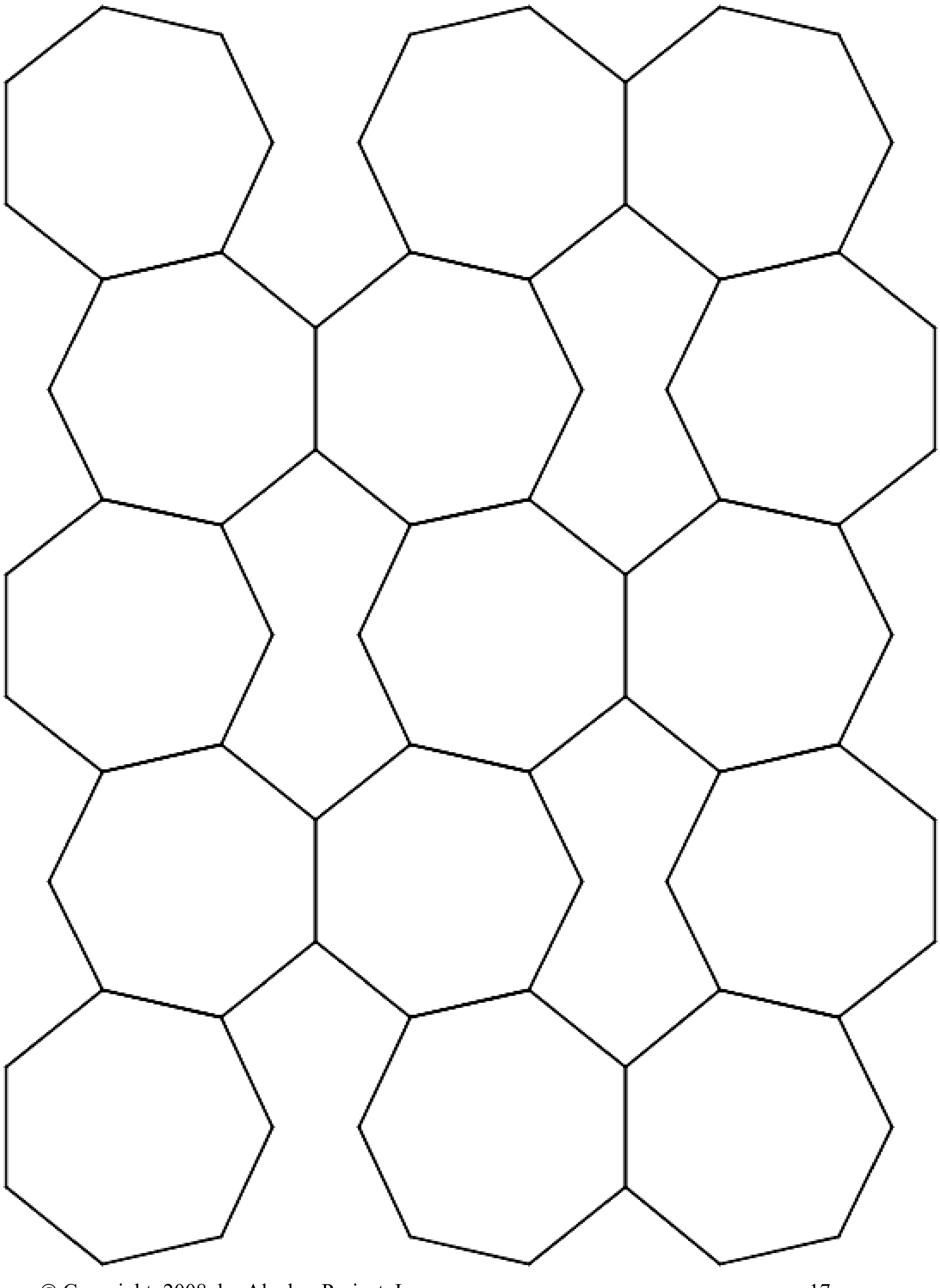
Cut along the solid lines and not along the dashed lines. Save the hexagon cut out from the center and replace it with a heptagon. Then use the removed hexagon to complete the surrounding of the heptagon. Put three of these together by overlapping the hexagons with the three dots. Tape the remaining heptagons into the surface in such a way that every heptagon is surrounded by seven hexagons and each hexagon is surrounded by there 3 heptagons alternating with 3 hexagons.

This is a picture of a finished hyperbolic soccer ball. This can be continued by adding more hexagons and heptagons in such a way that each heptagon is surrounded by seven hexagons and each hexagons surrounded by three heptagons interweaved with three 3 hexagons. See an example of this in the following photo taken in New Orleans.



Template for soccer ball and hyperbolic soccer ball. Cut along solid lines only.





NE3-2. Use ribbons to explore intrinsic straight lines (geodesics) on the hyperbolic soccer ball. You can also carefully stretch the surface and yarn between the two points. Or use the small toy car. What do you find?

Can you find two geodesics that do not intersect?
Can you make a large triangle (with geodesic sides)? What can you say about the sum of the angles of this triangle?
Compare intrinsic straight lines on the hyperbolic soccer ball to those on the plane and to those on the sphere.
What else to you notice?

Each group reports back to the whole class.